Justify your answers and show all your work. Unless otherwise indicated simplification of answers is not required.

1. Consider the following function

$$f(x) = -2x + 4\sqrt{x} - \frac{1}{x^2} + \pi.$$

4 marks (a) Compute the derivative of the function $f(x)$.

$$f(x) = -2x + 4x^{1/2} - x^{-2} + \pi$$

$$f'(x) = -2 + \frac{4}{2}x^{-1/2} + 2x^{-3} + 0$$

$$= -2 + 2x^{-1/2} + 2x^{-3}$$

4 marks (b) Compute the general anti-derivative of the function $f(x)$.

Let $F(x)$ be such that $F'(x) = f(x)$

Then,

$$F(x) = -x^2 + 4 \cdot \frac{2}{3} x^{3/2} + x^{-1} + \pi x + C.$$

4 marks 2. (a) Compute

$$\int_1^2 \frac{2-x}{\sqrt{x}}\,dx.$$

$$\int_1^2 \frac{2-x}{\sqrt{x}}\,dx = \int_1^2 \left(\frac{2}{\sqrt{x}} - \frac{x}{\sqrt{x}}\right) dx = \int_1^2 \left\{2x^{-1/2} - x^{1/2}\right\} dx$$

$$= 4x^{1/2} - \frac{2}{3}x^{3/2} \Big|_{x=1}^{2}$$

$$= 4\sqrt{2} - \frac{2}{3}(2)^{3/2} - \left(4 - 2/3\right)$$

$$= 4\sqrt{2} - \frac{2}{3}(2)^{3/2} - 4 + 2/3.$$

4 marks (b) Find a function $g(x)$ satisfying $g(\pi) = 1$ such that

$$g'(x) = \sin x + \cos(4x) + e^x.$$

$$g(x) = -\cos x + \frac{1}{4}\sin(4x) + e^x + C.$$

Set,

$$1 = g(\pi) = -\cos(\pi) + \frac{1}{4}\sin(4\pi) + e^\pi + C.$$

$$= 1 + 0 + e^\pi + C.$$

$$\Rightarrow \quad 1 = 1 + e^\pi + C$$

$$C = -e^\pi$$

$$\Rightarrow \quad g(x) = -\cos x + \frac{1}{4}\sin(4x) + e^x - e^\pi.$$

5 marks 3. Find the equation of the tangent line to

$$f(x) = \frac{\cos(2x)}{x}$$

at the point $x = \pi$.

Apply QR:

$$f'(x) = \frac{-\sin(2x)\cdot\frac{1}{2}x - \cos(2x)}{x^2}.$$

$$f'(\pi) = \frac{-\frac{\pi}{2}\sin(2\pi) - \cos(2\pi)}{\pi^2}$$

$= -1/\pi^2 \leftarrow$ slope of the tangent.

Use slope-point form

$$y - y_1 = m(x - x_1)$$

$m = -1/\pi^2$

$x_1 = \pi$

$y_1 = f(\pi) = \frac{\cos(2\pi)}{\pi} = \frac{1}{\pi}.$

So,

$$y - \frac{1}{\pi} = \frac{-1}{\pi^2}(x - \pi)$$

is the equation of the desired tangent line.

5 marks 4. Find the derivative of

$$f(x) = xe^{2x}\sin(x^2).$$

We apply product rule twice.
(alternatively, use tripple product rule)

$$f'(x) = (x)'e^{2x}\sin(x^2) + x\left(e^{2x}\sin(x^2)\right)'$$

$$= e^{2x}\sin(x^2) + x\left(2e^{2x}\sin(x^2) + e^{2x}\cdot 2x\cos(x^2)\right)$$

$$= e^{2x}\sin(x^2) + 2xe^{2x}\sin(x^2) + 2x^2e^{2x}\cos(x^2).$$

5. Compute the following integrals.

[5 marks] (a)

$$\int \frac{e^x}{(e^x+1)^2}\,dx$$

Let $u = e^x + 1$ $\quad du = e^x dx$

$$\int \frac{1}{(e^x+1)^2} e^x dx = \int \frac{1}{u^2} du = \frac{-1}{u} + C$$

$$= \frac{-1}{e^x+1} + C.$$

[5 marks] (b)

when $x = 0$
$u = 0$
when $x = \pi/2$
$u = 1$.

$$\int_0^{\pi/2} \sin x \cos x \; dx$$

Let $u = \sin x$, $\quad du = \cos x \, dx$

$$\int_{x=0}^{x=\pi/2} \sin x \cos x \, dx = \int_{u=0}^{u=1} u\,du = \frac{1}{2}u^2 \Big|_{u=0}^{u=1}$$

$$= \frac{1}{2}.$$

5 marks (c)

$$\int x^2 \ln x \, dx$$

Let $u = \ln x$ and $dv = x^2 dx$

then $du = \frac{1}{x} dx$ and $v = \frac{x^3}{3}$.

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^3 \cdot \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C.$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C.$$

5 marks 6. A spherical snow ball is melting such that its surface area is decreasing at a rate of $0.5\text{cm}^2/\text{min}$. How fast is the volume decreasing when the radius is 6cm? The Volume and Surface Area of a sphere are given by

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2$$

respectively. We seek $\frac{dV}{dt}$ and have $\frac{dA}{dt}$.

We first, however, find $\frac{dr}{dt}$!

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r}\frac{dA}{dt}.$$

Now,
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$= 4\pi r^2 \frac{1}{8\pi r}\frac{dA}{dt}.$$
$$= \frac{r}{2}\frac{dA}{dt}.$$

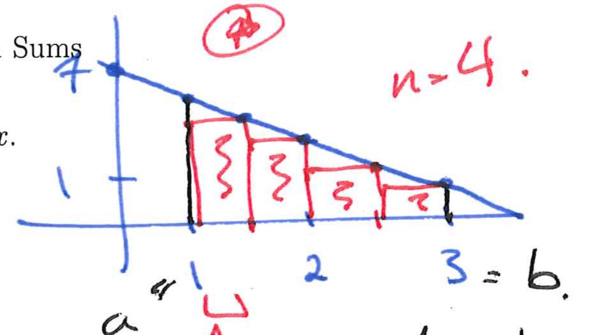
Want $\frac{dV}{dt}$ @ $r = 6\text{cm}$, $\frac{dA}{dt} = 1/2\ \text{cm}^2/\text{min}$.

$$\frac{dV}{dt} = \frac{6\text{cm}}{2}\cdot\frac{1}{2}\text{cm}^2/\text{min}$$
$$= \frac{3}{2}\text{cm}^3/\text{min}.$$

4 marks 7. (a) Approximate the following integral using Riemann Sums

$$\int_1^3 (-2x+7)\,dx.$$

Use right endpoints and $n = 4$ (ie. four bars).



$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 1/2.$

$\int_1^3 (-2x+7)\,dx \approx \sum_{i=1}^{4} f(x_i)\,\Delta x$

$x_1 = 1.5,\ x_2 = 2,\ x_3 = 2.5,\ x_4 = 3.$

$\sum_{i=1}^{4} f(x_i)\Delta x = f(1.5)\frac{1}{2} + f(2)\frac{1}{2} + f(2.5)\frac{1}{2} + f(3)\frac{1}{2}$

$= 4\cdot\frac{1}{2} + 3\cdot\frac{1}{2} + 2\cdot\frac{1}{2} + 1\cdot\frac{1}{2}$

$= 2 + \frac{3}{2} + 1 + \frac{1}{2}$

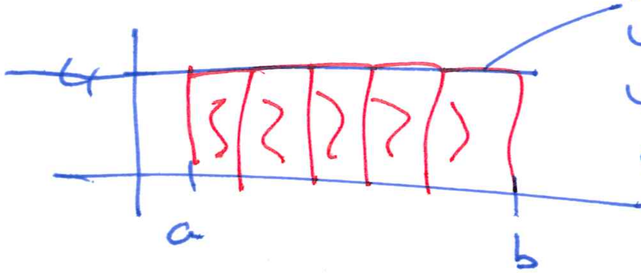
$= 3 + 2 = 5.$

2 marks (b) Is your approximation less than, greater than, or exactly equal to the true value of the integral? Explain why.

Less than. Observe the graph 4.
Since the function is decreasing and we used right endpoints we have missed some area.

3 marks (c) Sketch the graph of a new function where an approximation with Riemann Sums is exactly equal to the area under the curve.

Consider the function $f(x) = 4$.
The constant function.



No matter how many bars we use our 'approximation' with Riemann Sums is exactly the area under the curve.

7 marks 8. The *rate of change* of the height of an elevator is give by

$$r(t) = te^t$$

in meters/second. If at $t = 0$ seconds the elevator is 1 meter off the ground, how high is the elevator after 2 seconds have passed?

We are interested in the height and have its rate of change.

$$h(t) = \int r(t)\,dt \qquad \text{Let } u = t \quad dv = e^t dt$$
$$\qquad\qquad\qquad\qquad du = dt \quad v = e^t$$

$$= \int te^t dt.$$

$$= te^t - \int e^t dt$$

$$= te^t - e^t + C.$$

We know, $h(0) = 1$ so

$$1 = h(0) = 0 \cdot e^0 - e^0 + C$$
$$= -1 + C$$
$$\Rightarrow C = 2.$$

Hence, $h(t) = te^t - e^t + 2.$

After 2 seconds we have,

$$h(2) = 2e^2 - e^2 + 2 \text{ meters high.}$$
$$= e^2 + 2.$$

4 marks 9. (a) Sketch the graph of a function $f(x)$ satisfying the following two properties:

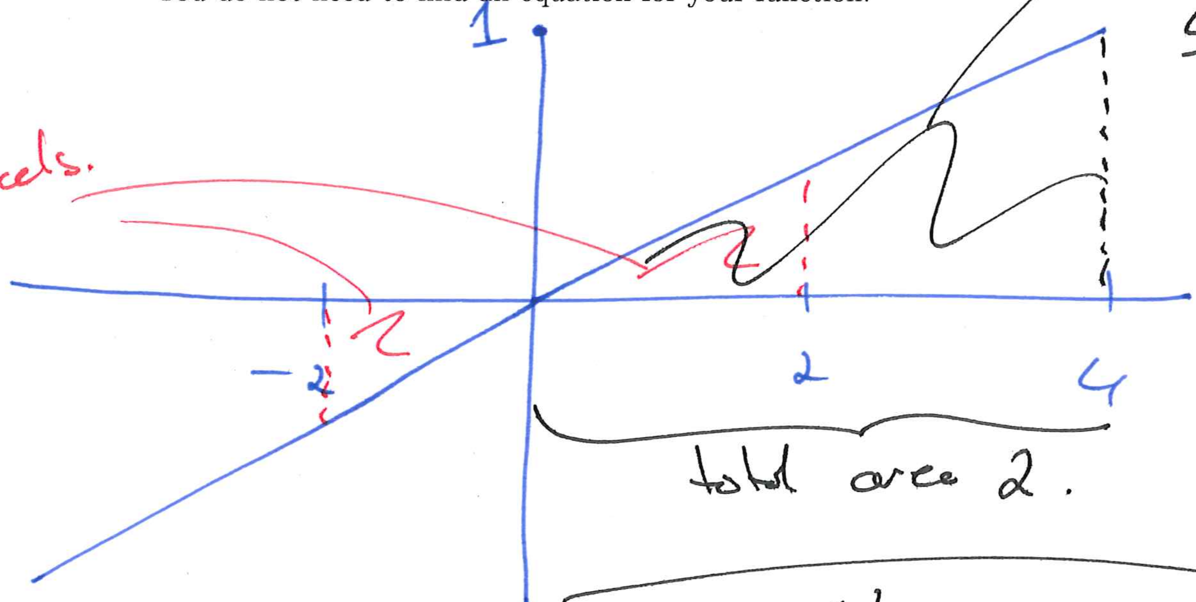
- $\int_{-2}^{2} f(x)dx = 0$
- $\int_{0}^{4} f(x)dx = 2$

You do not need to find an equation for your function.

There are many examples. One is:

$\dfrac{\text{base} \cdot \text{height}}{2} = \dfrac{4 \cdot 1}{2} = 2.$

area cancels.



4 marks (b) Find two values of b such that

$$\int_{-\pi}^{b} \sin(2x)\, dx = 0.$$

Ensure you justify your answer fully.

Solution #1

Compute:

$$\int_{-\pi}^{b} \sin(2x)dx = \left.\frac{-1}{2}\cos(2x)\right|_{x=-\pi}^{x=b}$$

$$= \frac{-1}{2}\cos(2b) + \frac{1}{2}\cos(-2\pi)$$

$$= \frac{1}{2} - \frac{1}{2}\cos(2b).$$

Need $\cos(2b) = 1$

Take, $b = 0$ and $b = \pi$.

In general

$2b = 2\pi n$

$b = \pi n, \quad n \in \mathbb{Z}.$

Solution #2

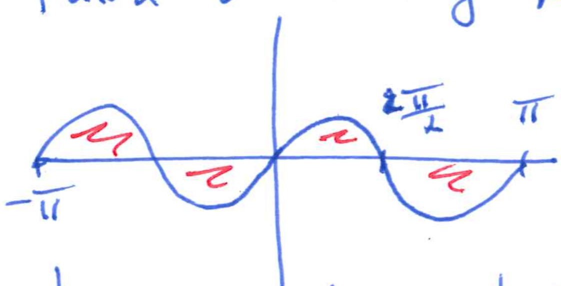
First of all $\int_{-\pi}^{-\pi} \sin(2x)dx = 0$

So $b = -\pi$ works.

Second, $\sin(2x)$ is odd so $\int_{-\pi}^{\pi} \sin(2x) = 0$

$b = \pi$.

Third draw a graph.



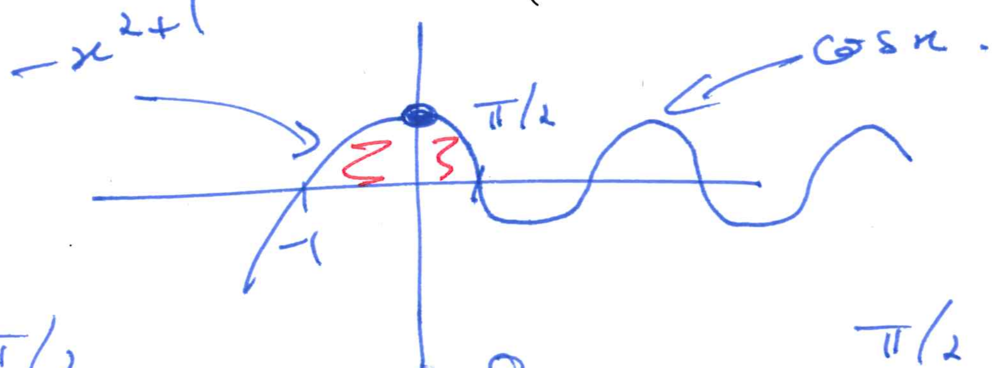
$b = 0$ or $b = \pi$ both work from the graph.

[5 marks] 10. Compute the integral

$$\int_{-1}^{\pi/2} f(x)dx$$

where

$$f(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ -x^2+1 & \text{if } x < 0 \end{cases}.$$



$$\int_{-1}^{\pi/2} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{\pi/2} f(x)\,dx$$

$$= \int_{-1}^{0} (-x^2+1)\,dx + \int_{0}^{\pi/2} \cos x\,dx$$

$$= -\frac{x^3}{3} + x \Big|_{-1}^{0} + \sin x \Big|_{0}^{\pi/2}$$

$$= \frac{-0^3}{3} + 0 - \left(\frac{1}{3} - 1\right) + \overset{=1}{\sin(\pi/2)} + \overset{=0}{\sin(0)}$$

$$= -\frac{1}{3} + 1 + 1$$

$$= -1/3 + 2 = 5/3 \quad //$$