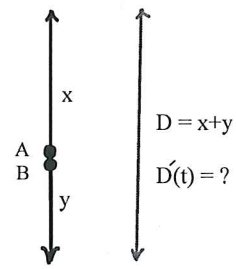
1. Two cyclists are on a north-south straight road. They both start from the same point on the road. Cyclist A rides north at a rate of 2 m/sec and 7 seconds later cyclist B starts riding south at 1 m/sec. At what rate is the distance separating the two cyclist changing 25 seconds after cyclist A starts riding her bike?

Solution:

The first step is drawing a diagram. The distance of the two cyclists from the point of start increases, so $x'(t) = 2, y'(t) = 1$ m/sec are both positive. Label the distance between the two cyclists D, in this case we have

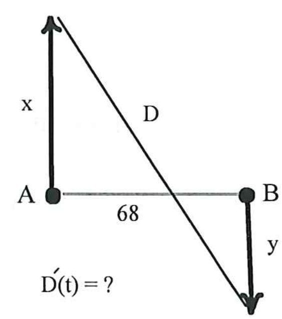


$$D = x + y \Rightarrow D'(t) = x'(t) + y'(t) = 2 + 1 = 3 \text{ m/sec}$$

Now suppose each cyclist is on a different road. Both road are parallel, straight and in the north-south direction and 68 metres apart. Both cyclists start at similar points on each road (that is, the cyclists are 68 metres apart at the start). Cyclist A rides north at a rate of 2 m/sec and 7 seconds later cyclist B starts riding south at 1 m/sec. At what rate is the distance separating the two cyclist changing 25 seconds after cyclist A starts riding her bike?

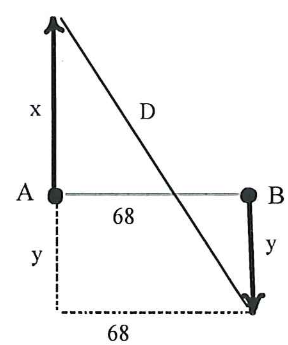
Solution:

This time the two cyclists are travelling on different roads, as given in the diagram on the right. We can make a right triangle for which $D'(t)$ is the unknown. The length of one leg of this triangle is fixed and equal to 68 m, the length of the other leg is the sum of the distances that each cyclist travels. By Pythagorean formula we have



$$D^2 = 68^2 + (x + y)^2 \Rightarrow 2DD' = 0 + 2(x + y)(x' + y')$$

First, we should find x, y and D. Cyclist A after 25 seconds has travelled $x(25) = 2 \times 25 = 50$ m and since cyclist B started 7 seconds later, the time for cyclist B is $25 - 7 = 18$ seconds, and the distance y at this moment is $y(18) = 1 \times 18 = 18$, so

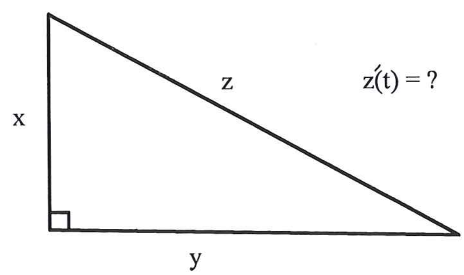


$$x + y = 50 + 18 = 68 \ , \ D = \sqrt{68^2 + 68^2} = 68\sqrt{2}$$

and $D' = \dfrac{(x + y)(x' + y')}{D} = \dfrac{68 \times 3}{68\sqrt{2}} = \dfrac{3}{\sqrt{2}}$ m/sec.

In a right triangle, the lengths of all sides are changing in such a way that the area of the triangle remains constant and is always equal to 6 m^2. Suppose x and y are the two legs (that is, the two sides that meet at a right angle) and z is the hypotenuse and x is increasing at the rate of 2 m/s. How fast is the hypotenuse changing when $x = 3$ m?

Solution: The first step is to draw a diagram and label the variables as instructed.



We know that the area of this right triangle is $A = 6 = \frac{1}{2}xy$ and also by Pythagorean formula we have $z^2 = x^2 + y^2$. The unknown is $z'(t)$, for which we need to differentiate both sides $2zz' = 2xx' + 2yy'$ and dividing by 2 gives

$$zz' = xx' + yy'$$

The given information is $x = 3$ m and $x'(t) = 2$ m/s. We need to find y, y' and z first. To find y and y' we use the area formula and its derivative:

$$A = \frac{1}{2}xy \Rightarrow 6 = \frac{1}{2}3y \Rightarrow y = 4$$

$$\text{differentiate } \Rightarrow 0 = \frac{1}{2}(x'y + xy') = \frac{1}{2}(2 \times 4 + 3 \times y') \Rightarrow y' = -\frac{8}{3}\, m/s$$

To find z, we use Pythagorean formula

$$z = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

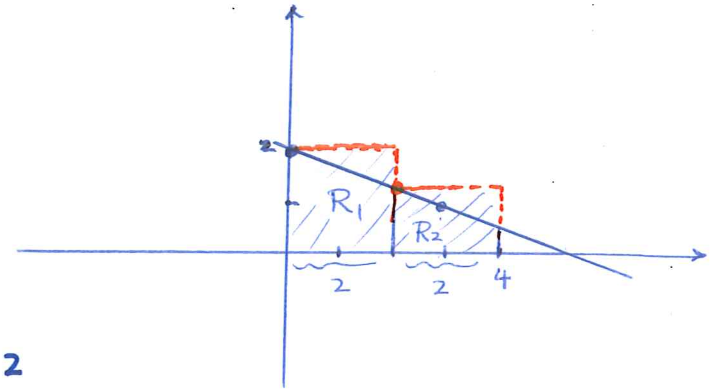
Now we are ready to find z':

$$z' = \frac{xx' + yy'}{z} = \frac{(3 \times 2) + (4 \times -\frac{8}{3})}{5} = -\frac{14}{15}\, m/s$$

(3)

$f(x) = -\frac{1}{3}x + 2$

x	y
0	2
3	1



a) $n = 2$: $\Delta x = \frac{4-0}{2} = 2$

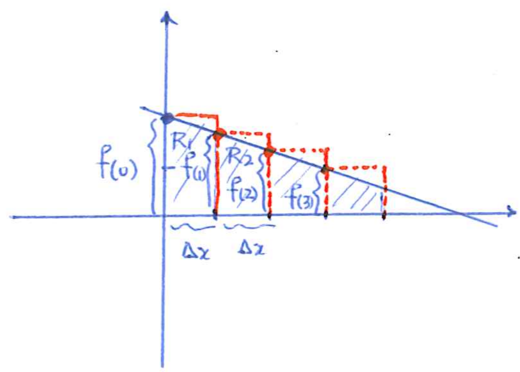
$R_1 = f(0) \cdot \Delta x = 2 \cdot 2 = 4$

$R_2 = f(2) \cdot \Delta x = \left(-\frac{1}{3} \cdot 2 + 2\right) \cdot 2$

$= \left(-\frac{2}{3} + 2\right) \cdot 2 = \frac{4}{3} \cdot 2 = \frac{8}{3}$

$\Rightarrow$ Area $\approx R_1 + R_2 = 4 + \frac{8}{3} = \frac{20}{3}$

(b) $n = 4$: $\Delta x = \frac{4-0}{4} = 1$



$R_1 = f(0) \cdot \Delta x = 2 \cdot 1 = 2$

$R_2 = f(1) \cdot \Delta x = \left(-\frac{1}{3} \cdot 1 + 2\right) \cdot \Delta x$

$= \frac{5}{3} \cdot 1 = \frac{5}{3}$

$R_3 = f(2) \cdot \Delta x = \left(-\frac{1}{3} \cdot 2 + 2\right) \cdot \Delta x = \frac{4}{3} \cdot 1 = \frac{4}{3}$

$R_4 = f(3) \cdot \Delta x = \left(-\frac{1}{3} \cdot 3 + 2\right) \cdot \Delta x = 1 \cdot 1 = 1$

$\Rightarrow$ Area $\approx R_1 + R_2 + R_3 + R_4 = 2 + \frac{5}{3} + \frac{4}{3} + 1 = 6$

(c) By integrating:

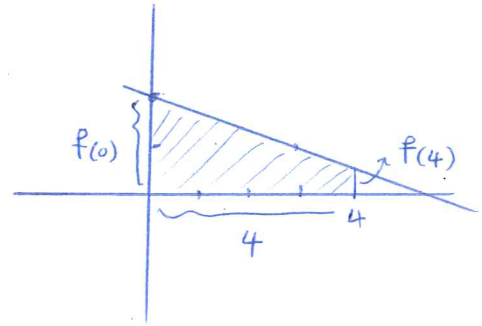
$$A = \int_0^4 \left(-\frac{1}{3}x + 2\right) dx = \left(-\frac{1}{3} \cdot \frac{1}{2} x^2 + 2x\right)\Bigg|_{x=0}^{x=4} = \underbrace{\left(-\frac{1}{6}(4)^2 + 2 \cdot 4\right)}_{F(b)} - \underbrace{(0+0)}_{F(a)}$$

$$= \frac{16}{3}$$

(HW4 solution)

By the picture:

Area of the trapzoid $= \dfrac{f(0)+f(4)}{2} \cdot 4$



$f(0) = 2$

$f(4) = -\frac{1}{3} \cdot 4 + 2 = \frac{2}{3}$

$$\Rightarrow A = \frac{2+\frac{2}{3}}{2} \cdot 4 = \frac{\frac{8}{3}}{2} \cdot 4 = \frac{8}{6} \cdot 4 = \frac{16}{3}$$

First approximation: $\frac{20}{3}$

our error from the actual area: $\frac{20}{3} - \frac{16}{3} = \frac{4}{3}$

Second approx: 6

error $= 6 - \frac{16}{3} = \frac{2}{3}$ → smaller error ⇒ Better approx.

Our approx. is an over-estimate. without doing the computation, we see that the graph is decreasing, so left Riemann sum will be giving $\underbrace{\text{rectangles, larger}}_{\text{with areas}}$ than the actual area.

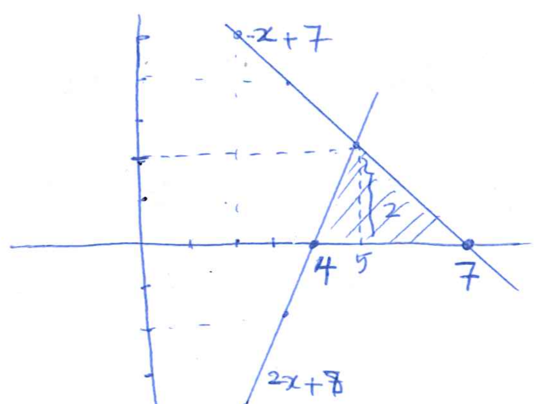
4)

$y_1 = -2(-x+1) - 6 = 2x - 8$

x	y
2	-4
3	-2

$y_2 = -x + 7$

x	y
2	5
3	4



x-intercepts for → $y_1 = 0 \Rightarrow x = 4$

↳ $y_2 = 0 \Rightarrow x = 7$

Area $= \frac{b \cdot h}{2} = \frac{3 \cdot 2}{2} = 3$

(a)

Two graphs cross when $2x - 8 = -x + 7$

$$\Rightarrow 3x = 15 \Rightarrow x = 5 \Rightarrow f(5) = 2(5) - 8 = 2$$

(c) $\int_4^7 h(x)\,dx \underset{\text{split from 5}}{=} \int_4^5 (2x-8)\,dx + \int_5^7 (-x+7)\,dx$

$h(x)$ has two lines

$$= \left(x^2 - 8x\right)\Big|_{x=4}^{x=5} + \left(-\frac{1}{2}x^2 + 7x\right)\Big|_{x=5}^{x=7}$$

$$= (5^2 - 40) - (4^2 - 32) + \left(\left(-\frac{1}{2}\cdot 7^2 + 49\right) - \left(-\frac{1}{2}\cdot 5^2 + 35\right)\right)$$

$$= (-15) - (-16) + \left(\left(-\frac{49}{2} + 49\right) - \left(-\frac{25}{2} + 35\right)\right)$$

$$= -15 + 16 + \frac{49}{2} - \frac{45}{2} = 3 \checkmark$$

5) $\int_{-\pi}^{\pi} \left(\sin x + 2x^4 + 2\right) dx = \left(-\cos x + 2\cdot\frac{1}{4+1}x^{4+1} + 2x\right)\Big|_{x=-\pi}^{x=\pi}$

$$= \left(-\cos\pi\right) - \left(\cos(-\pi)\right) + \left(\frac{2}{5}\pi^5 - \frac{2}{5}\underbrace{(-\pi)^5}_{-\pi^5}\right) + \left(2\pi - 2(-\pi)\right)$$

$$= +1 - 1 + \frac{2}{5}\pi^5 + \frac{2}{5}\pi^5 + 2\pi + 2\pi$$

$$= \frac{4}{5}\pi^5 + 4\pi$$