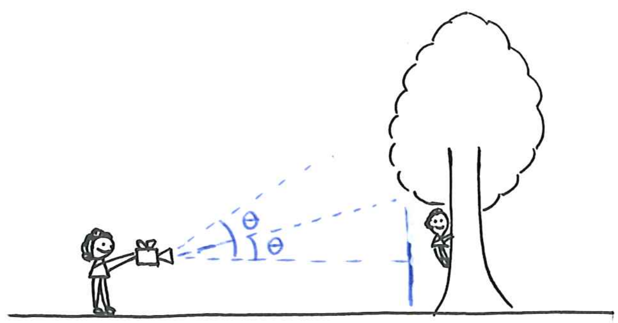
One more Related Rates problem.

Your lumberjack friend is climbing a tree at a constant speed of 0.5 m/s. You stand 6 meters from the tree and film with your camera. How fast is the camera's angle changing when your friend is 6 m up.




* changing quantities:

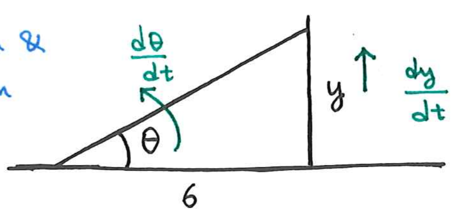
→ distance of your friend from the ground : $y$

→ angle between the camera & the ground : $\theta$

* fixed quantity:

→ your distance from the tree

(1) Diagram & Notation




(2) Information:
$$\begin{cases} \dfrac{dy}{dt} = y'(t) = 0.5 \text{ m/s} \\ \dfrac{d\theta}{dt} = \theta'(t) = \,? \quad \text{when } y = 6 \text{ m} \end{cases}$$

(3) Relate the variables: Use Trigs ⇝ sin, cos or tan ?

$y$ is opposite to $\theta$ and 6 m distance is adjacent so:

$$\tan \theta(t) = \frac{y(t)}{6}$$

(4) Differentiate both sides:

Do we need the Quotient rule? Not really because $\frac{y}{6} = \frac{1}{6} \cdot y$ ⇝ Constant multiple

$$\underbrace{\tan}_{\text{outside}}(\underbrace{\theta(t)}_{\text{inside}}) = \frac{1}{6}\, y(t)$$

* Recall : $\big(\underbrace{f}_{\text{out}}(\underbrace{g(x)}_{\text{in}})\big)' = f'(g(x)) \cdot g'(x)$

$$\big(\tan(\theta(t))\big)' = (\tan \oslash)' = \underbrace{(1+\tan^2 \oslash)}_{\text{outside derivative}} \cdot \underbrace{\oslash'}_{\text{inside derivative}}$$

$$(1 + \underbrace{\tan^2}_{\text{find this}} \theta(t)) \cdot \underbrace{\theta'(t)}_{\text{unknown}} = \frac{1}{6}\, \underbrace{y'(t)}_{\text{given}}$$

(5) Substitute the info and solve for the unknown $\theta'$ ?

At this moment $y = 6$ so $\tan\theta = \frac{6}{6} = 1$

$$\Rightarrow \quad 1 + \tan^2\theta = 1 + 1^2 = 2$$

$$\Rightarrow \quad 2 \cdot \theta'(t) = \frac{1}{6} \times 0.5 \quad \Rightarrow \quad \theta'(t) = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{24} \text{ rad/sec}$$
$$= 2.4 \text{ deg/sec}$$

* Note : You could use $(\tan\theta(t))' = \sec^2\theta \cdot \theta'(t)$ as well. this gives you the same answer but with one extra step in computation.

$\sec\theta = \frac{1}{\cos\theta}$ so what is $\cos\theta$ ?

$$\cos\theta = \frac{\text{adjacent}}{\text{hyp}} = \frac{6}{\sqrt{6^2+6^2}} = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}}$$

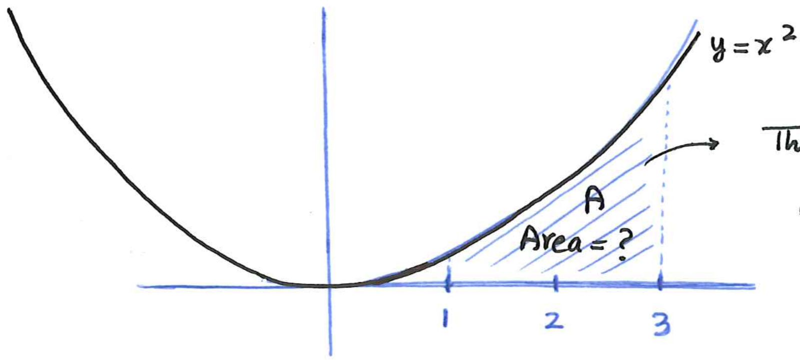
$$\Rightarrow \sec\theta = \frac{1}{\cos\theta} = \sqrt{2} \quad \Rightarrow \quad \sec^2\theta = 2.$$

which is exactly equal to $1+\tan^2\theta = 1$ as we calculated above.

MATH 190 : Part I : Differential Calculus ✓

Part II : Integral Calculus

**Example 1.** We want to find the area under the curve $y = x^2$ when $x$ is between 1 and 3.



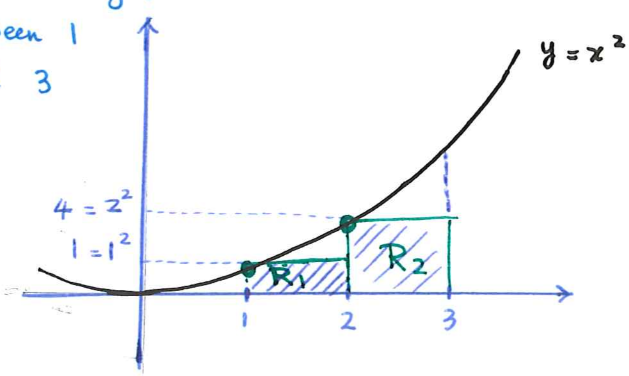

This is not a shape whose area has a formula. We need other methods

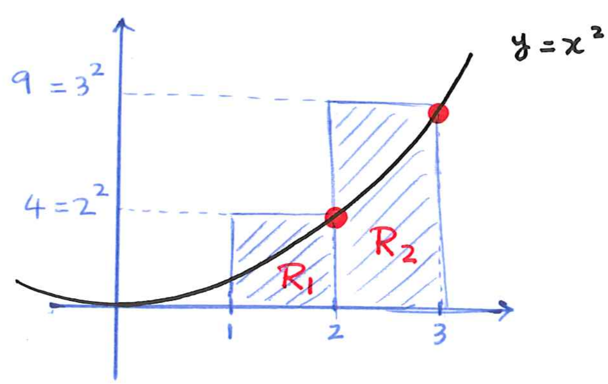
**Question:** How do I find this area ?

Approximate the area with rectangles

Recall : h b → Area = base . height = b x h

2 rectangles between 1 and 3




**Attempt 1:** In each sub interval use the left endpoint i.e. $x = 1$ and $x = 2$

$R_1$ = height x base = $1 \times 1 = 1$

$R_2 = 1 \times 4 = 4$

Actual area $\approx$ $R_1 + R_2 = 1 + 4 = 5$

approx.

**Attempt 2:** Make 2 rectangles with Right endpoints in each subinterval i.e. $x = 2$ & $x = 3$

$R_1$ = base x height = $1 \times 4 = 4$

$R_2 = 1 \times 9 = 9$

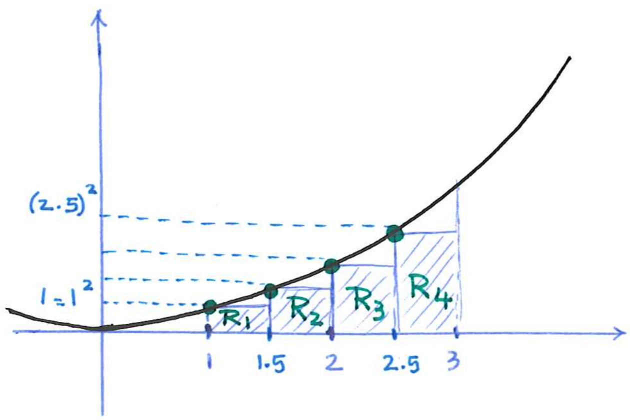
Actual area $\approx R_1 + R_2 = 4 + 9 = 13$

Problem with this approximation : It is not accurate $\rightsquigarrow$ crude approx.

We only get that $5 < A < 13$

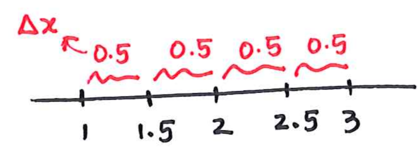
How to make the approx. more accurate ?

Use more rectangles $\rightsquigarrow$ 4 rectangles between $x=1$ and $x=3$.




4 rectangles in $[1,3]$ :

$\Delta x$ = base length $= \frac{3-1}{4} = \frac{2}{4} = 0.5$




- 4 rectangles with left end-points : $x = 1, 1.5, 2, 2.5$

$R_1$ = base x heigh = $\Delta x$ x function value at left end points

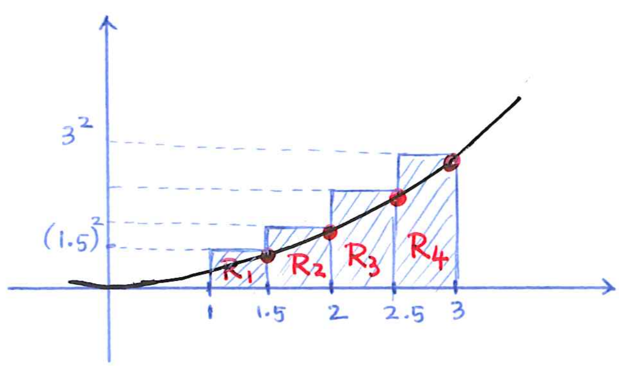
$\Rightarrow R_1 = 0.5 \times 1^2 = 0.5$

$R_2 = 0.5 \times (1.5)^2 = 1.125$

$R_3 = 0.5 \times (2^2) = 2$

$R_4 = 0.5 \times (2.5)^2 = 3.125$

Actual area $\approx R_1 + R_2 + R_3 + R_4$
$= 0.5 + 1.125 + 2 + 3.125 = \boxed{6.75}$




Better approx:

$6.75 <$ Actual area $< 10.75$

- 4 rectangles with right end points : $x = 1.5, 2, 2.5, 3$

$R_1 = 0.5 \times (1.5)^2 = 1.125$

$R_2 = 0.5 \times (2)^2 = 2$

$R_3 = 0.5 \times (2.5)^2 = 3.125$

$R_4 = 0.5 \times 3^2 = 4.5$ $\Rightarrow$ Actual area $\approx R_1 + R_2 + R_3 + R_4 = \boxed{10.75}$