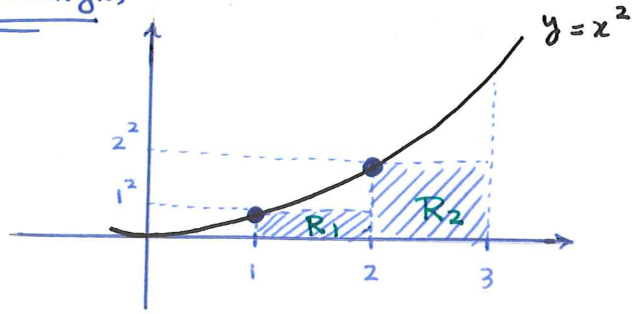
Lecture 25
Nov 2

Last Class :

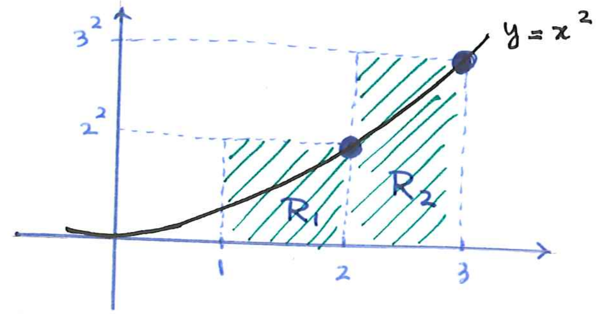
Find the area under the curve $y = x^2$ between $x = 1$ and $x = 3$ by approximation with rectangles.

2 rectangles



left end-point in each subinterval

$R_1 + R_2 = 5$



right end-point

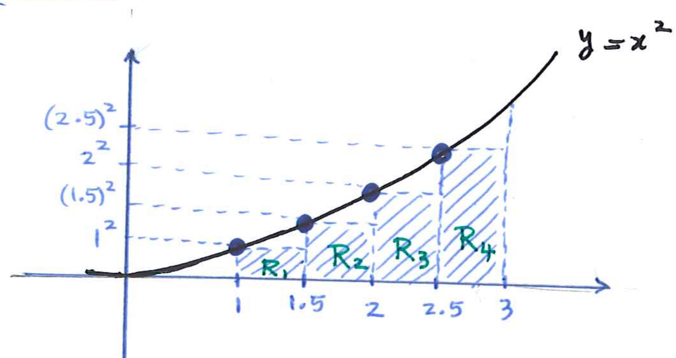
$R_1 + R_2 = 13$

So the actual area under the curve : A

$$5 < A < 13$$

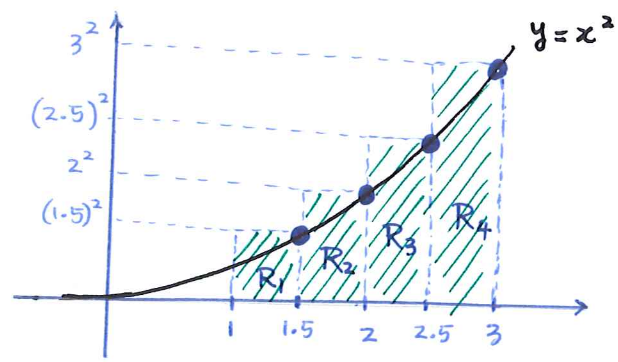
Make a Better Approx.

4 rectangles :



left end-point in each sub interval

$R_1 + R_2 + R_3 + R_4 = 6.75$



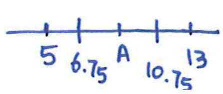
right endpoint

$R_1 + R_2 + R_3 + R_4 = 10.75$

Actual area

$$6.75 < A < 10.75$$

* As you see the approximation interval is getting closer to the actual area A.

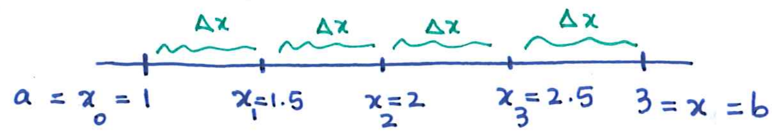


Let's formulate this procedure :

We are given an interval $\underset{\uparrow\ \uparrow}{[1, 3]}$ $[a, b]$ and we want to split this interval into 4 equal subintervals that are the base of each rectangle.

Let's denote the endpoints of each of these subintervals by

$x_0, x_1, x_2, \ldots$



We then have :

$$\Delta x = \frac{3-1}{4} = \frac{b-a}{4} = 0.5$$

Choosing the left endpoint in each subinterval, the height of the rectangle comes from the y-value at the left end-point so:

$$\left.\begin{aligned} R_1 &= f(x_0)\cdot \Delta x \\ R_2 &= f(x_1)\cdot \Delta x \\ R_3 &= f(x_2)\cdot \Delta x \\ R_4 &= f(x_3)\cdot \Delta x \end{aligned}\right\} \Rightarrow A \approx R_1 + R_2 + R_3 + R_4$$

$$= f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x$$

$$= \boxed{\sum_{i=0}^{3} f(x_i)\cdot \Delta x}$$

→ Short form for summing the areas.

Sigma notation

i varies from 0 to 3

1^{st} point : x_0 , last point: x_3

Now choose the right endpoint in each subinterval :

$$A \approx R_1 + R_2 + R_3 + R_4 = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x$$

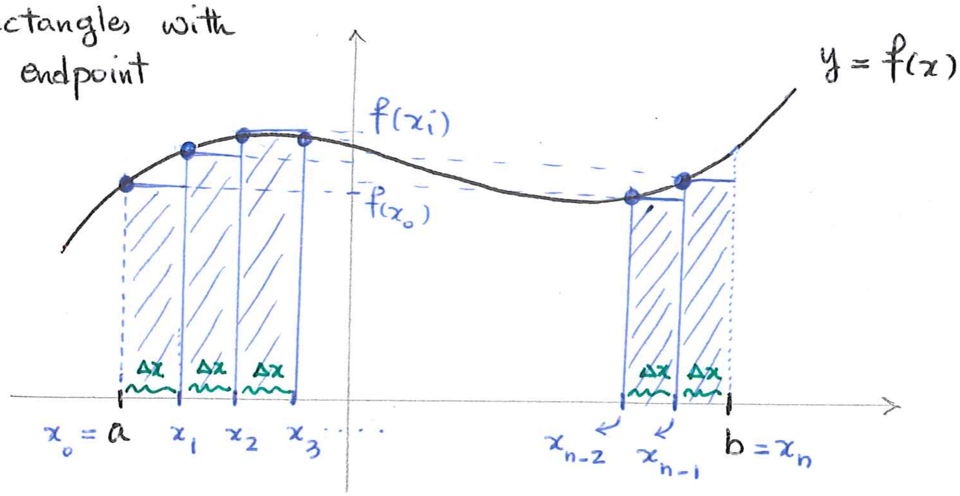
$$= \boxed{\sum_{i=1}^{4} f(x_i)\cdot \Delta x}$$

i starts from 1 and goes to 4

In general, we can do this process for any function $y = f(x)$ with any number of rectangles, say n:

$$\left.\begin{array}{l} \#\text{ of rectangles} = n \\ \text{interval} = [a, b] \end{array}\right\} \Rightarrow \Delta x = \frac{b-a}{n}$$

* Make rectangles with the left endpoint



Sum of the areas $= f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + \cdots + f(x_{n-1}) \cdot \Delta x$

$$= \boxed{\sum_{i=0}^{n-1} f(x_i) \cdot \Delta x} \rightarrow \text{Left Riemann Sum}$$

* Make rectangles with the right end point

Sum $= f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x$

$$= \boxed{\sum_{i=1}^{n} f(x_i) \cdot \Delta x} \rightarrow \text{Right Riemann Sum}$$

To make approximation close to the actual area, we increase the number of rectangles, n, so Δx becomes smaller

Therefore, as $n \to \infty$, Sum of the areas approach to the actual area (both left and right Riemann sums will become the actual area.)

Mathematical Notation:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$$

$$= \text{Actual area}$$

Notation for

Area under the curve of $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\, dx$$

Thus:

$$\boxed{\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x}$$

Pieces of the notation:

$$\int_a^b \underbrace{f(x)}_{\text{"integrand"}}\, dx$$

analogous for small Δx

integrating with respect to x

integral bounds

"integrand" we want the area under $f(x)$

integral notation

$S \rightsquigarrow \int$

stretched S for Sum

Clicker Q : What is $\int_2^4 3 \, dx$?

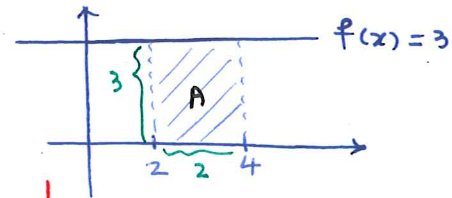
→ means : area under $f(x) = 3$ from $x = 2$ to $x = 4$:

A. 3

B. 6

C. 0

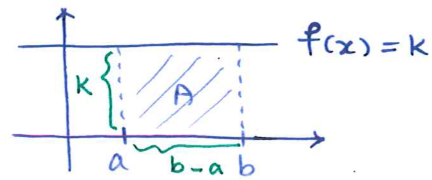
D. 12



$$A = 3 \cdot (4-2) = 3 \cdot 2 = 6$$

In general :

$$\boxed{\int_a^b k \, dx = k \cdot (b-a)}$$



Clicker Q : Suppose we know that

$$\int_0^3 x^2 \, dx = 9 .$$

What is $\int_0^3 2x^2 \, dx$

→ Double the y-values

⇒ Double the height of the rectangles

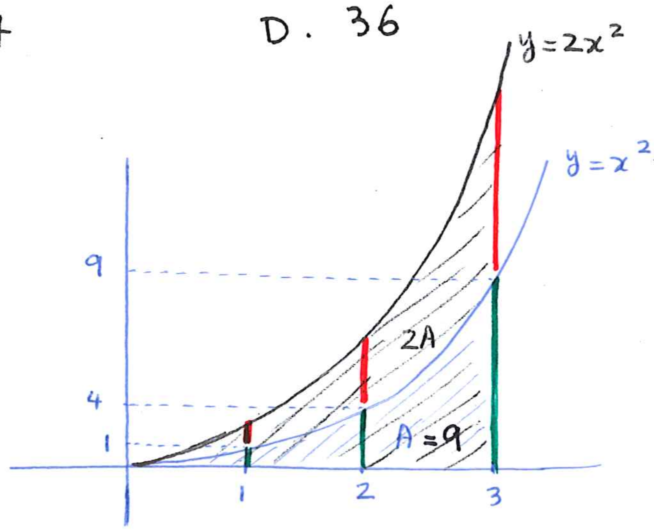
⇒ Area gets doubled

A. 9

B. 27

C. 18

D. 36



$$\int_0^3 2x^2 \, dx = 2 \int_0^3 x^2 \, dx$$

$$= 2 \cdot 9 = 18$$

In general :

$$\boxed{\int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx}$$

Clicker Q · What about $\int_0^3 (x^2+2)\,dx$

A. 11 C. 12

B. 15 D. 18

$$\int_0^3 (x^2+2)\,dx = \int_0^3 x^2\,dx + \int_0^3 2\,dx$$

$$= 9 + 2\cdot(3-0)$$

$$= 9 + 6 = 15$$

In general :

$$\boxed{\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$$