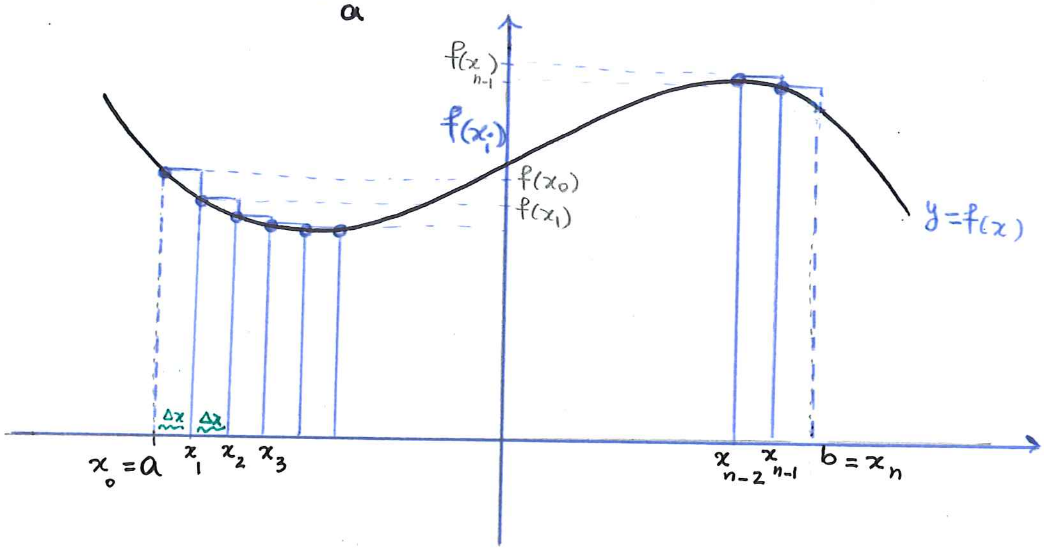
Last Class : Meaning of

$$\int_a^b f(x)\,dx$$

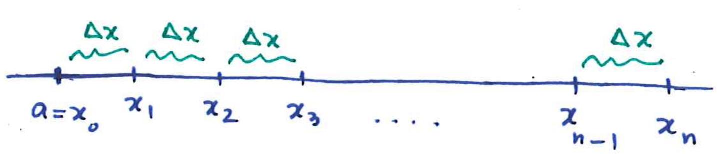


$\int_a^b f(x)\,dx \xrightarrow{\text{meaning}}$ Area under the curve of $y=f(x)$ from $x=a$ to $x=b$.

$\hookrightarrow$ Mathematical Definition :

Consider the interval $[a,b]$. We want to divide the interval into "n" rectangles with equal width Δx so $\Delta x = \dfrac{b-a}{n}$.

This gives us the points on the x-axis, hence subintervals are



$$\Delta x = \frac{b-a}{n} \quad \text{or} \quad \Delta x = x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$$

Now depending on whether we choose the left or right endpoint in each subinterval we get :

Sum the areas of the rectangles using the left end point

$$f(x_0).\Delta x + f(x_1).\Delta x + f(x_2).\Delta x + \cdots + f(x_{n-1}).\Delta x$$

$$= \boxed{\sum_{i=0}^{n-1} f(x_i).\Delta x} \longrightarrow \text{Left Riemann Sum}$$

using the right endpoint :

$$f(x_1).\Delta x + f(x_2).\Delta x + f(x_3).\Delta x + \cdots + f(x_n).\Delta x$$

$$= \boxed{\sum_{i=1}^{n} f(x_i).\Delta x} \longrightarrow \text{Right Riemann sum}$$

Now, to have the actual area we need to consider many rectangles which means $n \longrightarrow \infty$, this makes Δx smaller and smaller.

As the number of rectangles grow, the approximated area gets closer and closer to the actual area so :

$$\boxed{\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i).\Delta x = \int_a^b \underbrace{f(x)}_{\text{"integrand"}}\, \underbrace{dx}_{\text{notation for small } \Delta x} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i).\Delta x}$$

integral notation

integral bounds or limits

"integrand"

notation for small Δx

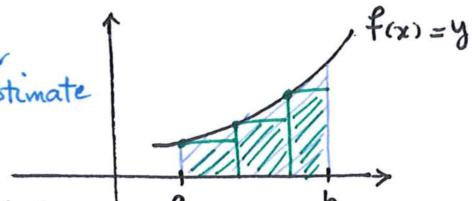
analogous to limit definition of the derivative.

as $n \to \infty$:

* both left and right Riemann sums become equal to each other and also equal to the actual area : $\int_a^b f(x)\,dx$.

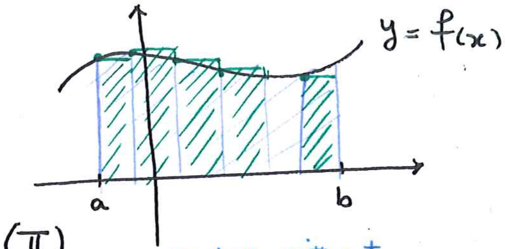
Clicker: Consider the following graphs :

under estimate



(I)

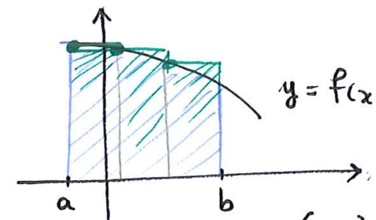
Less than the actual area



(II)

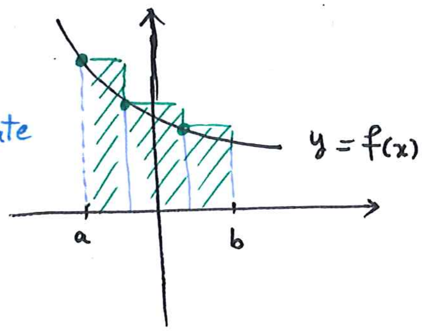
Can NOT tell without computation

Larger than the actual area

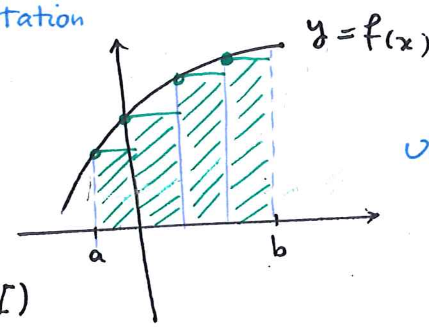


Over-estimate (III)

Over-estimate



(IV)



Under-estimate

(V)

Without doing any computation, find the graph(s) for which the left Riemann Sum is an underestimate of the actual area: $\int_a^b f(x)\,dx$

A. (I), (II), (IV)

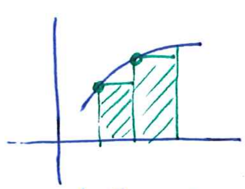
C. (III), (IV), (V)

$\boxed{\text{B}}$. (I), (V)

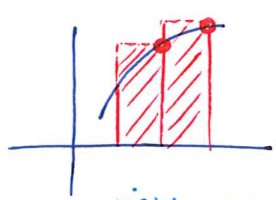
D. (III), (IV)

If we have an always increasing or decreasing graph then we can predict under and over estimate :

increasing



left end-point

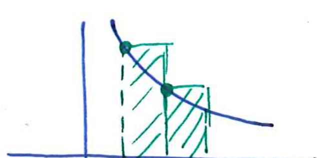


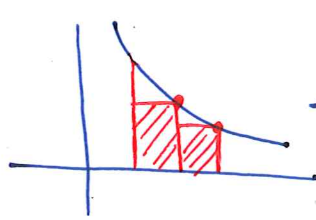
right end-point

left end-point : underestimate

right end-point : over estimate

decreasing





left : Overestimate

right : Underestimate

Some Integral Rules

- $\displaystyle\int_a^b \underbrace{K}_{\text{constant}}\, dx = K(b-a) \rightsquigarrow \int_3^5 -4\, dx = -4(5-3) = -8$

- $\displaystyle\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx \xrightarrow{\int_0^2 x^2 dx = \frac{8}{3}} \int_0^2 5x^2 dx = 5\int_0^2 x^2 dx$

  $= 5 \cdot \frac{8}{3} = \frac{40}{3}$

- $\displaystyle\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

  $\displaystyle\rightsquigarrow \int_0^2 (x^2+5)\, dx = \int_0^2 x^2 dx + \int_0^2 5\, dx$

  $= \frac{8}{3} + 5 \cdot (2-0) = \frac{8}{3} + 10$

Question: How do we find that

$$\int_0^2 x^2\, dx = \frac{8}{3} \quad ???$$

The hard and long way:

Use the limit definition of the integral:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

Does NOT matter which Riemann Sum we pick:

Let's pick the right one

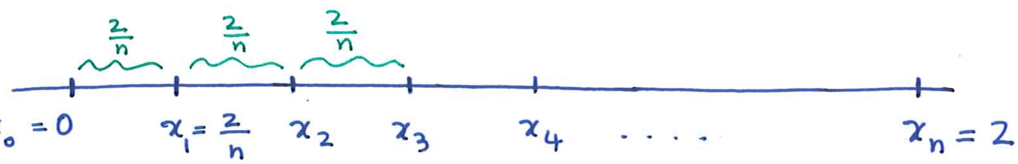
$$\int_0^2 x^2\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

We need to find a, b, x_i, $f(x_i)$ & Δx :

$$\int_0^2 x^2\,dx \quad : \quad \begin{matrix} a = 0 \\ b = 2 \end{matrix} \Rightarrow \Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

* Note : n is changing, it is not a fixed number $n \to \infty$

$\Delta x = \frac{2}{n}$ ⇝ steps in each subinterval :



$$\Rightarrow \quad x_0 = 0$$

$$x_1 = \frac{2}{n}$$

$$x_2 = 2 \cdot \frac{2}{n}$$

$$x_3 = 3 \cdot \frac{2}{n}$$

$$\vdots$$

$$x_{100} = 100 \cdot \frac{2}{n}$$

$$\vdots$$

$$x_i = i\,\frac{2}{n} = \frac{2i}{n}$$

$$x_i = \frac{2i}{n}$$

and $f(x) = x^2$ so

$$f(x_i) = \left(\frac{2i}{n}\right)^2$$

$$\boxed{\int_0^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n}}$$

$$= \ldots$$

we do algebra to simplify the summation and finding the limit. (You don't need to do this part!)

$$= \frac{8}{3}.$$

What is the short way?

$$\boxed{\text{Fundamental Theorom of Calculus}}$$

Link between "Derivative" & "Integral"

Idea: Derivatives & Integrals are opposite or inverse operations.

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where $F'(x) = f(x)$

The Function $F(x)$ is called the anti-derivative of $f(x)$.

Let's start with a simple example:

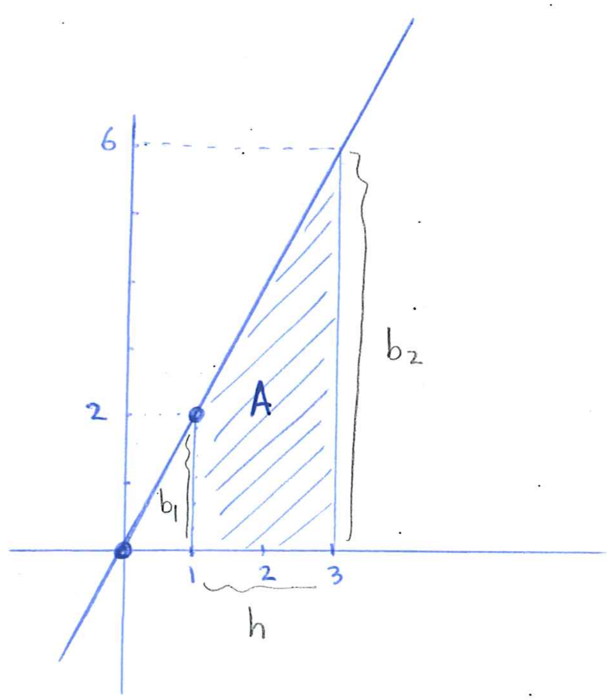
__Example 1__: Compute $\displaystyle\int_1^3 2x\,dx$

(a) using graph of the function and the area under that.

(b) using FTC.

(a) plot $y = 2x$

x	y
0	0
1	2



Find A:

either as a trapzoid or as area between two triangles:

$$A_{\text{trapzoid}} = \frac{(\text{longer base} + \text{shorter base}) \cdot \text{height}}{2}$$

$$= \frac{1}{2}(b_1 + b_2) \cdot h$$

$$= \frac{1}{2}(2+6) \cdot 2 = 8$$

Or $\text{Area of big triangle} = \frac{1}{2} bh = \frac{1}{2} \cdot 3 \cdot 6 = 9$

$A \text{ of smaller triangle} = \frac{1}{2} \cdot 1 \cdot 2 = 1$ $\Rightarrow$ $\text{Area in between} = 9 - 1 = 8$

(b) $\displaystyle\int_1^3 2x \, dx$

Find a function $F(x)$ such that $F'(x) = 2x$

One option: $F(x) = x^2$ $\rightsquigarrow$ antiderivative of $2x$

So $\displaystyle\int_1^3 2x \, dx = F(3) - F(1) \underset{F(x)=x^2}{=} 3^2 - 1^2 = 8$

Notation :

$$\int_1^3 2x \, dx = x^2 \Big|_{x=1}^{x=3} = 3^2 - 1^2 = 8$$

In general :

$$\int_a^b f(x) \, dx = \int_a^b F'(x) \, dx = F(x) \Big|_{x=a}^{x=b} = F(b) - F(a).$$