- HW 4 posted. Due: Wednesday, Nov 14
- Pick up your marked quizzes and HW from MLC

Nov 7
Lecture 27

To find $\int_a^b f(x)\,dx$, we find the anti-derivative of $f(x)$ first; i.e. $F(x)$ such that $F'(x) = f(x)$ and we evaluate the values $F(b)$ and $F(a)$ then we use

FTC: $$\boxed{\underbrace{\int_a^b f(x)\,dx}_{} = F(b) - F(a)}$$

This is called the definite integral of $f(x)$, because the lower and upper bounds of integral i.e. a & b are given

Examples:

(1) $\int_0^2 x^2\,dx = F(2) - F(0) \overset{F(x)=\frac{x^3}{3}}{=} \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$

Find a function $F(x)$ such that:

$$F'(x) = x^2 \rightsquigarrow F(x) = \frac{x^3}{3} = \frac{1}{3}x^3$$

* You may start by $F(x) = x^3$ then derive $F'(x) = 3x^2 \rightsquigarrow$ NOT quite right

divide by 3 $F(x) = \frac{x^3}{3}$ so $F'(x) = \frac{1}{3}\cdot 3x^2 = x^2$

shortcut notation for $F(3) - F(-1)$

(2) $\int_{-1}^{3} x^3\,dx = \boxed{\frac{1}{4}x^4 \Big|_{x=-1}^{x=3}} = \frac{1}{4}(3)^4 - \frac{1}{4}(-1)^4$

$= \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$

we want: $(F(x))' = x^3$ so $F(x) = \frac{1}{4}x^4$

* Put brackets for negative values substitute.
* There is a $-$ from FTC, Don't forget to multiply that in the term following that

Power rule for integration

In general;

$$\boxed{\text{If } f(x) = x^n \text{ then } n \neq -1 \quad F(x) = \frac{1}{n+1}x^{n+1}}$$

$\rightarrow$ Think: Is this the "only" $F(x)$?

(3) $\displaystyle\int_1^2 \left(\sqrt{x} + 3x^5 - \frac{1}{\sqrt[3]{x^2}}\right) dx$

First rewrite the integrand in power form, and break the integral into 3 integrals:

$$\int_1^2 x^{\frac{1}{2}}\,dx + \int_1^2 3x^{5}\,dx - \int_1^2 x^{-\frac{2}{3}}\,dx$$

$$= \left(\frac{1}{\frac{1}{2}+1}\, x^{\frac{1}{2}+1}\Big|_{x=1}^{x=2}\right) + 3\cdot\left(\frac{1}{5+1}\, x^{5+1}\Big|_{x=1}^{x=2}\right) - \left(\frac{1}{-\frac{2}{3}+1}\, x^{-\frac{2}{3}+1}\right)\Big|_{x=1}^{x=2}$$

be careful!

* Recall:

$\displaystyle\int_a^b \left(f(x) \pm g(x)\right) = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\displaystyle\int_a^b \left(k\, f(x)\right) dx = k \int_a^b f(x)\,dx$

$\displaystyle\int_a^b x^n\,dx = \frac{1}{n+1}\, x^{n+1}\Big|_{x=a}^{x=b}$

Simplify then Use FTC & substitute the bounds

$$= \frac{1}{\frac{3}{2}}\, x^{\frac{3}{2}}\Big|_{x=1}^{x=2} + \cancel{3}\cdot\frac{1}{\underset{2}{\cancel{6}}}\, x^{6}\Big|_{x=1}^{x=2} - \frac{1}{\frac{1}{3}}\, x^{\frac{1}{3}}\Big|_{x=1}^{x=2}$$

$$= \left(\underbrace{\frac{2}{3}\cdot 2^{\frac{3}{2}}}_{F(2)} - \underbrace{\frac{2}{3}\, 1^{\frac{3}{2}}}_{F(1)}\right) + \left(\underbrace{\frac{1}{2}\cdot 2^{6}}_{F(2)} - \underbrace{\frac{1}{2}\cdot 1^{6}}_{F(1)}\right) - \left(\underbrace{3\cdot 2^{\frac{1}{3}}}_{F(2)} - \underbrace{3\cdot 1^{\frac{1}{3}}}_{F(1)}\right)$$

$$= \frac{2}{3}\cdot 2^{\frac{3}{2}} - \frac{2}{3} + 32 - \frac{1}{2} - 3\cdot 2^{\frac{1}{3}} + 3$$

* No need to simplify unless the question asks us to.

* Be extra careful with negative signs when evaluating the anti-derivative at the endpoints. Note that FTC gives a − sign that should be considered in algebra.

<u>Question</u> What if $n = -1$?

$$\int_1^2 x^{-1}\, dx =$$

Let's start with power rule : $\int_1^2 x^{-1}\, dx = \frac{1}{-1+1} x^{-1+1} \Big|_{x=1}^{x=2} = \frac{1}{0} x^0 \Big|_{x=1}^{x=2} = !!!$

When the power is (-1), we need something different :

What function $F(x)$ has its derivative equal to $x^{-1} = \frac{1}{x}$?

Remember : $F(x) = \ln x \Rightarrow F'(x) = \frac{1}{x}$

So $\int_1^2 x^{-1}\, dx = \int_1^2 \frac{1}{x}\, dx = \ln x \Big|_{x=1}^{x=2} = \ln 2 - \underbrace{\cancel{\ln 1}}_{0} = \ln 2$

<u>Example</u> :

<u>Clicker Q</u> : $\int_0^1 e^x\, dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$

A . 1 $\boxed{\text{C}}$. $e - 1$

B . e D . 0 E. No idea

For what $F(x)$ we have $(F(x))' = e^x$? $\Rightarrow F(x) = e^x$

Comes from FTC

What about $\int_0^1 e^{-x}\, dx = -e^{-x} \Big|_{x=0}^{x=1} = \left(-e^{-1}\right) - \left(-e^{0}\right) = -e^{-1} + 1$

Comes from $F(x)$ Comes from $F(x)$

find $F(x)$ such that $F'(x) = e^{-x}$.

inside

Is $F(x) = e^{-x}$ OK ? Let's check : $F'(x) = \left(e^{-x}\right)' = e^{-x} \cdot -1 = -e^{-x}$

outside $(e^{⊘})' = e^{⊘} \cdot ⊘'$ NOT quite right !

Multiply by -1 and it'll work :

$F(x) = -e^{-x} \Rightarrow F'(x) = -e^{-x} \cdot -1 = e^{-x}$ ✓

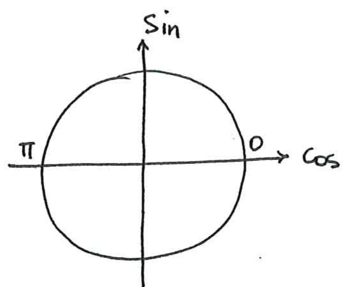
Example: $\displaystyle\int_0^{\pi} \sin x \, dx =$

What to choose for $F(x)$ so that $F'(x) = \sin x$?

Does $F(x) = \cos x$ work ? Check: $F'(x) = (\cos x)' = -\sin x$

So choose $F(x) = -\cos x$ then $F'(x) = -(-\sin x) = \sin x$ ✓

$$\Rightarrow \int_0^{\pi} \sin x \, dx = -\cos x \Big|_{x=0}^{x=\pi}$$

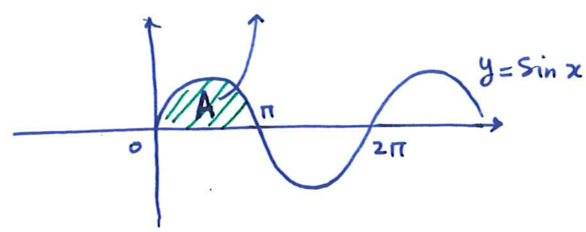


Cos 0 = 1 & Cos π = −1

$$= \left(-\overset{-1}{\cos \pi}\right) - \left(-\overset{1}{\cos 0}\right) = (-(-1)) - (-1)$$

$$= 1 + 1 = 2$$

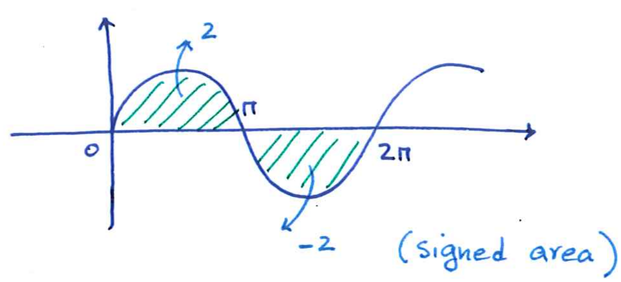
Graphically $\displaystyle\int_0^{\pi} \sin x \, dx$ means :



So $A = 2$

What about

$$\int_0^{2\pi} \sin x \, dx = 0$$



By formula:

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_{x=0}^{x=2\pi} = \left(-\overset{1}{\cos 2\pi}\right) - \left(-\overset{1}{\cos 0}\right)$$

$$= -1 + 1 = 0$$