Indefinite integral

Recap:

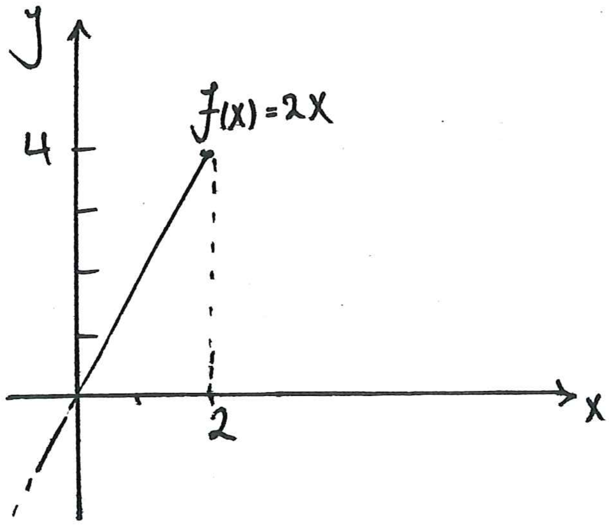
To find $\int_0^2 2x\,dx$, we can:

1) Use the definition (the hard way!):

Choose left/right Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

2) Find the area under the graph $y = f(x)$:



$$\int_0^2 2x\,dx = \text{Area of triangle}$$

$$= \frac{4\times 2}{2} = 4$$

3) Use the Fundamental Theorem of Calculus (FTC):

FTC roughly says, integeration and differentiation are opposite each other!

To use FTC:

Find a function, $F(x)$, such that $F'(x) = f(x)$.

We call $F(x)$ an anti-derivative of $f(x)$.

Then FTC says:

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

In our example, $f(x) = 2x$.

Take $F(x) = x^2$, then we see $F'(x) = (x^2)' = 2x = f(x)$

So FTC implies:

$$\int_0^2 2x\,dx = \int_0^2 (x^2)'\,dx \overset{\text{FTC}}{=} (2)^2 - (0)^2 = 4$$

But is x^2 the only anti-derivative of $2x$?

How about $x^2 + 8$? Check: $(x^2+8)' = (x^2)' + \underbrace{(8)'}_{0} = 2x$ ✓

How about $x^2 + \pi^e$? Check: $(x^2+\pi^e)' = 2x$ ✓

The general anti-derivative for $f(x) = 2x$ is given by

$$F(x) = x^2 + C \text{ where } C \text{ is some constant}$$

Indefinite integral $\longleftrightarrow$ The general anti-derivative

Indefinite integral: $\int 2x\,dx = x^2 + C$ for C some constant

Definite integral: $\int_0^2 2x\,dx = 4$

Another example:

$$\int x^2\,dx = \frac{1}{3}x^3 + C \text{ where } C \text{ is a constant}$$

Check: $\left(\frac{1}{3}x^3 + C\right)' = \frac{1}{3}(3x^2) + 0 = x^2$ ✓

Also

$$\int e^x\,dx = e^x + C \text{ for some } C \text{ constant}$$

Check: $(e^x + C)' = (e^x)' + (C)' = e^x + 0 = e^x$

Does this Constant mess up "using FTC to find definite integrals"?

Same Constant

$$\int_0^2 2x\,dx = \int_0^2 (x^2+4)'\,dx$$

$$\text{FTC} = x^2+4\Big|_0^2 = ((2)^2+4)-((0)^2+4)$$

$$= 4+4-0-4 = 4 \checkmark$$

No matter C is, the answer is always the same!

So, FTC works for any anti-derivative.

Important remark:

When using FTC to solve definite integrals, you can take any anti-derivative (usually $C=0$)

But be careful, when solving an indefinite integral your answer must have Constant C.

General anti-derivatives for some elementary functions:

n ≠ -1, n is a rational number

- $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$ where C is a constant
- $\int e^x \, dx = e^x + C$ ——— " ———
- $\int \sin x \, dx = -\cos x + C$ ——— " ———
- $\int \cos x \, dx = \sin x + C$ ——— " ———
- $\int \frac{1}{x} \, dx = \ln|x| + C$ ——— " ———

absolute value as "ln" only takes positive values

Also, as before:

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

and

$$\int k f(x) \, dx = k \int f(x) \, dx$$ where k is a constant.

Example 1:

Find the general anti-derivative of

$$f(x) = 9e^x + \frac{5}{\sqrt{x}} - 2\cos x + 5$$

$$\int f(x)\,dx = \int 9e^x + 5x^{-\frac{1}{2}} - 2\cos x + 5 \; dx$$

$$= \int 9e^x\,dx + \int 5x^{-\frac{1}{2}}\,dx + \int -2\cos x\,dx + \int 5\,dx$$

$$= 9\int e^x\,dx + 5\int x^{-\frac{1}{2}}\,dx - 2\int \cos x\,dx + 5\int x^0\,dx$$

one Constant is enough!

$$= 9(e^x + C) + 5\left(\frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1}\right)$$ where C is a Constant

$$-2(\sin x) + 5\left(\frac{1}{0+1} x^1\right)$$

Do not need to put different Constant for each indefinite integral, as sum of Constants is just another Constant

just another constant say D

$$= 9e^x + 9C + 5(2x^{\frac{1}{2}}) - 2\sin x + 5x^1$$

$$= 9e^x + 10x^{\frac{1}{2}} - 2\sin x + 5x + D$$

where D is a Constant.

Example 2:

Use your answer to example 1 to find

$$\int_0^{\pi} 9e^x + \frac{5}{\sqrt{x}} - 2\cos x + 5 \, dx$$

by FTC $\leftarrow$ $= 9e^x + 10x^{\frac{1}{2}} - 2\sin x + 5x + D \Big|_0^{\pi}$

$$= \left[9e^{\pi} + 10\sqrt{\pi} - 2\underbrace{\sin(\pi)}_{} + 5(\pi) + D\right]$$

$$- \left[9\underbrace{e^0}_{1} + 10\underbrace{\sqrt{0}}_{0} - 2\underbrace{\sin(0)}_{0} + \underbrace{5(0)}_{0} + D\right]$$

$$= 9e^{\pi} + 10\sqrt{\pi} - 0 + 5\pi + D$$

$$- \left[9 + 0 - 0 + 0 + D\right]$$

$$= 9e^{\pi} + 10\sqrt{\pi} + 5\pi - 9$$

Example 3:

et $f(x) = \frac{1}{2} \sin x + \sqrt{3} \cos x$.

Find $F(x)$ such that $F'(x) = f(x)$ and $F(0) = 2$.

$$\int f(x)\,dx = \int \frac{1}{2} \sin x + \sqrt{3} \cos x \, dx$$

$$= \frac{1}{2} \int \sin x \, dx + \sqrt{3} \int \cos x \, dx$$

$$= \frac{1}{2}(-\cos x + C) + \sqrt{3}(\sin x) \quad C \text{ is a Constant}$$

$$= -\frac{1}{2} \cos x + \sqrt{3} \sin x + P \quad \text{where } P \text{ is a Constant}$$

$$= F(x)$$

$\star\, F(0) = 2$

$$F(0) = -\frac{1}{2} \overbrace{\cos(0)}^{1} + \sqrt{3} \overbrace{\sin(0)}^{0} + P = \frac{-1}{2} + P \overset{(\star)}{=} 2$$

$$\Rightarrow P = 5/2$$

$$F(x) = -\frac{1}{2} \cos x + \sqrt{3} \sin x + 5/2$$