Last Week :

→ Definite Integral :

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = \boxed{F(b) - F(a)}$$ ⇝ This is a number.

Find an anti-derivative $F(x)$ such that $F'(x) = f(x)$

FTC : $\int$ and $'$ cancel each other.

→ Indefinite Integral :

$$\int f(x)\,dx = \boxed{F(x) + C}$$

This is a function.

⇝ A function $f(x)$ can have many anti-derivatives $F(x)$, but they're all different by a constant term.

* for any Constant C :

$$\left(F(x) + C\right)' = F'(x) = f(x)$$

Clicker Q. Find the general anti-derivative of

$$f(x) = \frac{x}{2} - e^x + \sin x - \frac{5}{\sqrt{x}} + 2$$

A. $F(x) = x^2 - e^x - \cos x - 10\sqrt{x} + 2 + C$

B. $F(x) = \frac{x^2}{4} - e^x + \cos x + 10\sqrt{x} + 2x$ $+C$

C. $F(x) = \frac{x^2}{4} - e^x - \cos x - 10\sqrt{x} + 2x + C$

D. $F(x) = 2x^2 - e^x - \cos x - 10\sqrt{x} + C$ $+2x$

term by term : $\frac{x}{2} \rightsquigarrow \frac{1}{2} \cdot \frac{1}{1+1} x^{1+1} = \frac{1}{4}x^2$, $e^x \rightsquigarrow e^x$, $\sin x \rightsquigarrow -\mathrm{Co}$

$\frac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}} \rightsquigarrow 5 \cdot \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} = 5 \cdot \frac{1}{\frac{1}{2}} x^{\frac{1}{2}} = 10\sqrt{x}$

<u>Clicker Q</u> : The general anti-derivative of $f(x) = (2x+1)^2$ is

$$F(x) = \frac{1}{6}(2x+1)^3 + C \; .$$

A. Yes

B. NO

How to verify ? Check if $F'(x) = f(x)$.

$$F(x) = \frac{1}{6} \blacksquare^3 + C$$

chain rule

$$F'(x) = \frac{1}{6} \cdot 3 \blacksquare^2 \cdot \blacksquare' + 0 = \frac{1}{6} \cdot 3 \cdot (2x+1)^2 \cdot 2$$

$$= \frac{6}{6}(2x+1)^2 = (2x+1)^2 \checkmark$$

<u>Question</u> : How should we compute

$$\int (2x+1)^2 \, dx$$

* We want to find anti-derivative of $(2x+1)^2$ by using integration techniqu

From the question above, we know that the answer should be

$F(x) = \frac{1}{6}(2x+1)^3 + C$ Now, let's see how we find it .

Easy case : $\int x^2 \, dx = \frac{1}{3} x^3 + C$

Let's write the integral in a form whose integral is easy to find :

Take $2x+1 = u$ then integral becomes $\int u^2 \, dx$

The variables in the integral must be all the same, so we need to chai

dx to du :

we have $2x + 1 = u$

$$\frac{d}{dx}(2x+1) = \frac{du}{dx} \xrightarrow[\text{respect to } x]{\text{Derive both sides with}} \quad \Rightarrow \quad 2 = \frac{du}{dx} \xrightarrow{\text{solve for } dx} dx = \frac{du}{2}$$

Now integral becomes : $\int u^2 \cdot \frac{du}{2} = \frac{1}{2} \int u^2 \, du$ → we know how to compu this

$$= \frac{1}{2} \cdot \frac{1}{3} u^3 + C$$

$$= \frac{1}{6} u^3 + C \underset{\text{go back to } x:}{=} \boxed{\frac{1}{6}(2x+1)^3 + C}$$

Example. $\int e^{5x}\, dx =$

We know how to compute $\int e^{x} dx$, so let's rewrite the integral with a u-substitution:

take $5x = u$ then $\frac{d}{dx}(5x) = \frac{du}{dx} \Rightarrow 5\,dx = du \Rightarrow dx = \frac{du}{5}$

$$\int e^{5x}\, dx = \int e^{u}\, dx = \int e^{u} \cdot \frac{du}{5}$$

$$= \frac{1}{5}\int e^{u}\, du = \frac{1}{5} e^{u} + C$$

$$= \frac{1}{5} e^{5x} + C$$

* This technique in integration is called $\boxed{\text{substitution}}$. We need to know which piece should be substi by u so that when we replace dx and simplify, we are left wi a relatively easier integral only in terms of u.

Example. $\int 3x \cos(x^2)\, dx =$

→ What substitution you would use ...

3 choices: A. $x^2 = u$

B. $\cos(x^2) = u$

C. $3x = u$

1st choice: $x^2 = u \Rightarrow 2x = \frac{du}{dx} \Rightarrow 2x\, dx = du \Rightarrow dx = \frac{du}{2x}$

Rewrite the integral: $\int 3x \cos(x^2)\, dx = \int 3\cancel{x} \cdot \cos(u) \cdot \frac{du}{2\cancel{x}}$

$$= \frac{3}{2}\int \cos u\, du$$ ⇝ easy to evaluate and only in term of u

$$= \frac{3}{2}\sin u + C$$

$$= \frac{3}{2}\sin(x^2) + C$$

What about other choices :

B. if $\cos(x^2) = u \Rightarrow \frac{d}{dx}\left(\cos(x^2)\right) = \frac{du}{dx}$

$$\overset{\text{chain}}{\Rightarrow} \quad -\sin(x^2)\cdot 2x = \frac{du}{dx}$$

$$\underset{dx}{\overset{\text{solve for}}{\Rightarrow}} \quad -\sin(x^2)\cdot 2x \; dx = du \Rightarrow dx = -\frac{du}{\sin(x^2).\, 2x}$$

Rewrite the integral : $\displaystyle\int 3x \cos(x^2)\, dx = \int 3x \cdot u \cdot \frac{-du}{\sin(x^2).2x}$

$$= \frac{-3}{2}\int \frac{u}{\sin(x^2)}\, du$$ ⇝ NOT an improvement from the original integral, still involves

C. If $3x = u \Rightarrow 3\, dx = du \Rightarrow dx = \frac{du}{3}$

$$\int 3x \cos(x^2)\, dx = \int u \cos(x^2)\, \frac{du}{3}$$

$$= \frac{1}{3}\int u \cos(x^2)\, du$$ ⇝ NOT an easier integral

Conclusion. There is NO fixed rule that tells us how to substitute. By practicing, we'll learn how to foresee one step ahead to know that what substitution will eventually simplifies all x's and we get an integral only in terms of u.

→ Hint : Choose the substitution, such that its derivative is someho existing in the integral.

Worksheet.

1) Evaluate the following indefinite integrals.

a) $\displaystyle\int \frac{x^2}{(1+x^3)^2}\,dx$

b) $\displaystyle\int \frac{\ln x}{x}\,dx$

c) $\displaystyle\int \sqrt{4-x}\,dx$

d) $\displaystyle\int (2x+5)(x^2+5x)^7\,dx$

e) $\displaystyle\int \tan x\,dx$

f) $\displaystyle\int \sin^2 x \cos x\,dx$

g) $\displaystyle\int \frac{\cos(5x)}{e^{\sin(5x)}}\,dx$

h) $\displaystyle\int x^2 e^{-4x^3}$

2) Evaluate the following definite integrals.

a) $\displaystyle\int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^3}\,dx$

b) $\displaystyle\int_{0}^{\pi} \cos x \cdot \sqrt{\sin x}\,dx$

c) $\displaystyle\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

d) $\displaystyle\int_{1}^{4} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$