Last Class :

Recall the main anti-derivatives :

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad \text{when } n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

* Remember the Constant of integration in indefinite integral

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

One Integration technique we learned lass class : $\boxed{\text{Substitution}}$

The original integral does NOT look like one of the above simpl integrals, but if we apply substitution ▨ = u and also find dx and replace them into the original and simplify, we'll get one of the above integrals in terms of u .

Example. Worksheet (1a) :

$$\int \frac{x^2}{(1+x^3)^2}\,dx =$$

Suitable substitution :

$u = 1 + x^3$

then

$du = 3x^2\,dx \Rightarrow dx = \frac{du}{3x^2}$

This appears in the integrand

Rewrite in term of u

$$\int \frac{\cancel{x^2}}{u^2} \cdot \frac{du}{3\cancel{x^2}} = \frac{1}{3}\int u^{-2}\,du$$

power rule $= \frac{1}{3} \cdot \frac{1}{-2+1} u^{-2+1} + C$

simplify $= -\frac{1}{3}u^{-1} + C$

go back to x. $= -\frac{1}{3}(1+x^3)^{-1} + C$

Clicker Q : $\int \frac{\ln x}{x}\,dx$, what u would you choose ?
(wsh 1b)

if
$u = \ln x$ ← A. $\ln x$

$\Rightarrow du = \frac{1}{x}\,dx$

appears in the integrand

B. $\frac{1}{x}$

C. $\frac{\ln x}{x}$

D. None of the above / No idea.

What about :

(wsh 1h) $\int x^2 e^{-4x^3}\,dx$ then $u = ?$

A. x^2

if
$u = -4x^3$ ← B. $-4x^3$

$du = -12x^2\,dx$

is there !

C. e^{-4x^3} → This also works.

D. Still No idea :(

• $\int \frac{\ln x}{x}\,dx = \int \frac{u}{x}\cdot x\,du = \int u\,du$

Rewrite

$u = \ln x$

$\Rightarrow du = \frac{1}{x}\,dx$

$\Rightarrow dx = x\,du$

$= \frac{1}{1+1}u^{1+1} + C$

$= \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\ln x)^2 + C}$

• $\int x^2 e^{-4x^3}\,dx = \int x^2 e^u \cdot \frac{du}{-12x^2} = -\frac{1}{12}\int e^u\,du$

Rewrite

$u = -4x^3$

$\Rightarrow du = -12x^2\,dx$

$\Rightarrow dx = \frac{du}{-12x^2}$

$= -\frac{1}{12}e^u + C$

$= \boxed{-\frac{1}{12}e^{-4x^3} + C}$

Use the other substitution for $\int x^2 e^{-4x^3}\,dx$

$$\boxed{\begin{aligned} u &= e^{-4x^3} \\ \Rightarrow du &= -12x^2 e^{-4x^3}\,dx \\ \Rightarrow dx &= \frac{du}{-12x^2 \underbrace{e^{-4x^3}}_{u}} \end{aligned}}$$

$$= \int \cancel{x^2}\,\cancel{u}\,\frac{du}{-12\cancel{x^2}\,\cancel{u}}$$

$$= -\frac{1}{12}\int du$$

$$= -\frac{1}{12}u + C$$

$$= -\frac{1}{12}e^{-4x^3} + C$$

Substitution in the case of definite integrals:

$$\int_0^1 x^2 e^{-4x^3}\,dx$$

There are two choices:

(1) transform the integral bounds to "u" as well and in the final step, you don't need to convert back to x, just apply FTC with the u-bounds.

$$\int_0^1 x^2 e^{-4x^3}\,dx$$

$u = -4x^3$

$\Rightarrow du = -12x^2\,dx$

$\Rightarrow dx = \dfrac{du}{-12x^2}$

Also transform the bounds.

lower bound: $x=0 \xrightarrow{u=-4x^3} u=-4(0)^3=0$

upper bound: $x=1 \longrightarrow u=-4(1)^3=-4$

$$= \int_0^{-4} -\frac{1}{12}e^u\,du = \left(-\frac{1}{12}\right)\,-\int_{-4}^{0} e^u\,du$$

$$= \frac{1}{12}e^u\Big|_{u=-4}^{u=0} = \frac{1}{12}\underbrace{e^0}_{1} - \frac{1}{12}e^{-4}$$

$$= \frac{1}{12} - \frac{1}{12}e^{-4}$$

(*) Note that we prefer to have lower bound smaller than the upper bound, if not we can easily swap the bounds but we should multiply by -1 :

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(2) To find the definite integral by substitution, we can ignore the bounds and find the general anti-derivative, then in the final step we convert back to the original variable x and we apply FTC with the x-bounds :

$$\int_0^1 x^2 e^{-4x^3}\,dx = -\frac{1}{12}\int e^u\,du = -\frac{1}{12}e^u + C$$

$$= -\frac{1}{12}e^{-4x^3} + C \Big|_{x=0}^{x=1}$$

$$= -\frac{1}{12}e^{-4(1)^3} + C - \left(-\frac{1}{12}e^0 + C\right)$$

$$= -\frac{1}{12}e^{-4} + \frac{1}{12}$$

Same answer as method (1).

(*) Recall that in definite integral, the constant C does N[illegible] matter since it'll be cancelled when FTC is applied.

Another Example:

$$\int_0^{\pi} \cos x \cdot \sqrt{\sin x}\, dx$$

$u = \sin x$

$\Rightarrow du = \cos x \, dx$

$\Rightarrow dx = \dfrac{du}{\cos x}$

bounds: $x = 0 \Rightarrow u = \sin 0 = 0$

$x = \pi \Rightarrow u = \sin \pi = 0$

$$= \int_0^0 \cos x \cdot \sqrt{u} \cdot \frac{du}{\cos x} = \int_0^0 \sqrt{u} \cdot du = 0$$

lower & upper bound are equal; this means NO area under the curve

$\Rightarrow \int = 0$

(*) Note: $\int_a^a f(x)\, dx = 0$

(*) These examples are all from the worksheet for substitution. Make sure you solve all the integrals on that worksheet.