HW 5 posted. Due date: Wednesday, Nov 28

Nov 21
Lecture 31

Integration by Parts

$$\boxed{\int u\, dv = uv - \int v\, du} \to \text{anti-product rule}$$

You are given an integral, you choose u and dv and you apply the above formula.

Example 1:

$$\int xe^x\, dx = \int u\, dv = uv - \int v\, du$$

Choose u and dv:

derive: $x = u$, $1 \cdot dx = du$

anti derive: $e^x dx = dv$, $e^x = v$

$$= xe^x - \int e^x\, dx$$

$$\boxed{= xe^x - e^x + C}$$

Let's check if we did it correctly:

$$(xe^x - e^x + C)' = \cancel{1 \cdot e^x} + x \cdot e^x \cancel{- e^x} = xe^x$$

Question: Does it matter how we choose u and dv?
What if I choose $u = e^x$ and $x\,dx = dv$?

$u = e^x$, $du = e^x \cdot dx$

$x' dx = dv$, $\frac{1}{1+1} x^{1+1} = v$, $\frac{1}{2} x^2 = v$

$$\Rightarrow \int xe^x dx = uv - \int v\, du$$

$$= e^x \cdot \frac{1}{2} x^2 - \underbrace{\int \frac{1}{2} x^2 e^x\, dx}_{\text{even harder than the original one.}}$$

X NOT a good choice

Usually our choice for "u" is prioritized by the following list:

(1) Logarithmic functions

(2) Algebraic functions such as polynomials

(3) trig functions

(4) exp functions.

Last Example: $x \rightsquigarrow$ algebraic function

$e^x \rightsquigarrow$ exp function $\rightsquigarrow x = u$

⊛ There are always exceptions to the list above.

Ex 2. $\int x \ln x \, dx$

$$\ln x = u \qquad x\,dx = dv$$

derive $\qquad\qquad$ anti derive

$$\frac{1}{x}\,dx = du \qquad \frac{1}{2}x^2 = v$$

$$\int x \ln x \, dx = \int u\,dv = uv - \int v\,du = \ln x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}\cancel{x^2}^{\,x} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

Remark: Note that IBP is different from substitution, even if we use "u". In substitution, we transform the integral $\int f(x)\,dx$ to a new integral $\int f(u)\,du$, so the variable is changing from x to u. However, in IBP we choose "u" and "dv" so that we can use the formula and simplify the integral, but the variable in the integral is still the original variable: x.

Clicker Q : $\int x \sin x \, dx$

How would you choose u and dv ?

A. $u = x$, $dv = \sin x \, dx$

B. $u = \sin x$, $dv = x \, dx$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$$\int x \sin x \, dx = \int u \, dv = uv - \int v \, du$$

$$= -x\cos x - \int -\cos x \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

Question : Find a particular function $F(x)$ whose derivative $F'(x) = x \sin x$ and $F(\frac{\pi}{2}) = 3$.

Definite integral with IBP ⇝ keep the integral bounds all along:

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Ex 4: $\displaystyle\int_0^{\pi/2} x^2 \sin x\,dx = \pi - 2$

$u = x^2 \qquad \sin x\,dx = dv$

$du = 2x\,dx \qquad -\cos x = v$

$$\int_0^{\pi/2} x^2 \sin x\,dx = \int_0^{\pi/2} u\,dv = uv\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} v\,du$$

$$= -x^2\cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x \cdot 2x\,dx$$

- $-x^2\cos x\Big|_0^{\frac{\pi}{2}} = -\left(\frac{\pi}{2}\right)^2 \cancel{\cos\frac{\pi}{2}}^{0} - \left(\cancel{-0^2}^{0}\cos 0\right) = 0$

- $2\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx = 2\left[\underset{u}{x}\,\underset{v}{\sin x}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx\right]$

another step with IBP

$x = u \qquad \cos x\,dx = dv$

$dx = du \qquad \sin x = v$

$$= 2\left[\frac{\pi}{2} - 1\right] = \boxed{\pi - 2}$$

- $x\sin x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}\underbrace{\cancel{\sin\frac{\pi}{2}}}_{1} - \underbrace{\cancel{0\sin 0}}_{0} = \frac{\pi}{2}$

- $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big|_0^{\frac{\pi}{2}} = -\cancel{\cos\frac{\pi}{2}}^{0} - \left(-\cancel{\cos 0}^{1}\right) = 1$

Practice Problems :

a) $\int 3xe^{-x}\,dx$

b) $\int \frac{\ln x}{x^2}\,dx$

c) $\int x^2 \cos x\,dx$

d) $\int \ln x \cdot 1\,dx$

e) $\int e^x \sin x\,dx$

f) $\int x^3 e^{x^2}\,dx$

g) $\int \sin x \cos x\,dx$

h) $\int \cos x \cdot \ln(\sin x)\,dx$

Definite Integrals

a) $\int_{-1}^{1} (2x+8)^3(-x+2)\,dx$

b) $\int_{0}^{\pi/2} (-6x+4)\cos x\,dx$

c) $\int_{1}^{e^3} x^7 \ln x\,dx$

d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2x \cos(3x+\pi)\,dx$

Practice Problems :

(d) $\int \ln x \, dx = uv - \int v \, du$

$u = \ln x \qquad 1 \cdot dx = dv$

$du = \frac{1}{x} dx \qquad x = v$

$= x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

(e) $\int e^{x} \sin x \, dx$

$u = \sin x \qquad e^{x} dx = dv$

Will complete it next class.