HW 5 Q4 and Q5 : typo fixed ⟶ Due date: Wed, Nov 28

Q4: $a(t) = -9$ Q5: $a(t) = -50000$

Nov. 23
Lecture 32

These are class notes. Neater notes will be posted soon.

Last Class : Integration by parts :

$$\int u\, dv = uv - \int v\, du$$

Definite Integral ⟶
$$\int_a^b u\, dv = uv \Big|_a^b - \int_a^b v\, du$$

Practice Problems from last class :

(1 e) $\int e^x \sin x\, dx$

$$\boxed{\begin{array}{ll} u = \sin x & e^x\, dx = dv \\ du = \cos x\, dx & e^x = v \end{array}}$$

$$\int e^x \sin x\, dx = uv - \int v\, du$$

$$= e^x \sin x - \underbrace{\int e^x \cos x\, dx}_{\text{another IBP}}$$

$$\boxed{\begin{array}{ll} u = \cos x & e^x\, dx = dv \\ du = -\sin x \cdot dx & e^x = v \end{array}}$$

$$= e^x \sin x - \left(uv - \int v\, du\right)$$

$$= e^x \sin x - \left(\cos x \cdot e^x - \int e^x \cdot -\sin x\, dx\right)$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

So we've got:

$$\underbrace{\int e^x \sin x\,dx}_{} = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{\text{solve for } \int e^x \sin x\,dx}$$

Are we stuck in a loop?! NOT really!

$$\int e^x \sin x\,dx + \int e^x \sin x = e^x \sin x - e^x \cos x$$

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$$

$$\boxed{\int e^x \sin x\,dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x} + C$$

* Remark: In this particular, it's equally easy if you take $u = e^x$ and $dv = \sin x\,dx$

$$-\frac{1}{2} e^{2\pi} + C = 2 \Rightarrow \boxed{C = 2 + \frac{1}{2} e^{2\pi}}$$

Suppose we want a particular anti-derivative of $e^x \sin x$, say $F(x)$ such that $\underbrace{F(2\pi) = 2}_{\text{helps to find a particular } C}$, then what is $F(x)$?

helps to find a particular C.

From above: $F(x) = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C$

$$F(2\pi) = 2$$

$$\Rightarrow F(2\pi) = \cancel{\frac{1}{2} e^{2\pi} \sin 2\pi}^{\,0} - \frac{1}{2} e^{2\pi} \cancel{\cos 2\pi}^{\,1} + C = 2$$

(1f) $\int x^3 e^{x^2}\,dx =$

$x^3 = u$ derive $\quad e^{x^2}\,dx = dv$ anti derive

$\cancel{x e^{x^2} = v}$

$\cancel{x e^{x^2} \cdot 2x = dv}$

BAD !

$u = e^{x^2} \quad x^3\,dx = dv$

$du = e^{x^2} \cdot 2x\,dx \quad \frac{1}{4}x^4 = v$

$$uv - \int \frac{1}{4}x^4 \cdot e^{x^2}\,2x\,dx$$

$$= uv - \frac{2}{4}\int x^5 e^{x^2}\,dx$$

BAD!

Substitution :

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

Rewrite the integral :

$$\int \underbrace{x^3}_{x^2 \cdot x} e^{x^2}\,dx = \int x^{\cancel{3}\,2} e^u \frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int x^2 e^u\,du$$

$$= \boxed{\frac{1}{2}\int u e^u\,du}$$

$$= \boxed{\frac{1}{2}ue^u - \frac{1}{2}e^u + C}$$

$$= \boxed{\frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C}$$

• $\frac{1}{2}\int t e^t\,dt$

IBP : $t = u \quad e^t dt = dv$

$1 \cdot dt = du \quad e^t = v$

$$\frac{1}{2}\int t e^t\,dt = \frac{1}{2}\int u\,dv = \frac{1}{2}\left(uv - \int e^t\,dt\right) = \frac{1}{2}te^t - \frac{1}{2}e^t + C$$

$\rightsquigarrow \frac{1}{2}ue^u - \frac{1}{2}e^u + C$

(2d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2x \cos(3x+\pi)\, dx$

$2x \cos(u) \dfrac{du}{3}$

Sub: $3x+\pi = u$

$3 \cdot 1\, dx = du$

$dx = \dfrac{du}{3}$ X

IBP:

$x = u$ $\qquad \cos(3x+\pi)\, dx = dv$

$dx = du$ $\qquad \int \cos(3x+\pi)\, dx = v$

$= \int \cos(u) \dfrac{du}{3}$

$= \dfrac{1}{3} \sin(u)$

$v = \dfrac{1}{3} \sin(3x+\pi)$

Substitution here:

$3x+\pi = u$

$3\, dx = du$

$dx = \dfrac{du}{3}$

$$\int x \cos(3x+\pi)\, dx = uv - \int v\, du$$

$$= x \frac{1}{3} \sin(3x+\pi) - \frac{1}{3}\int \sin(3x+\pi)\, dx$$

$$= \frac{1}{3} x \sin(3x+\pi) - \frac{1}{3} \underbrace{\int \sin(3x+\pi)\, dx}_{\bullet}$$

• $\displaystyle\int \sin(3x+\pi)\, dx = \int \sin(u) \frac{du}{3} = -\frac{1}{3}\cos(u) + C$

$= -\dfrac{1}{3} \cos(3x+\pi) + C$

Sub: $3x+\pi = u$

$3dx = du$

$dx = du/3$

$$\int x \cos(3x+\pi)\,dx = \frac{1}{3}x\sin(3x+\pi) + \frac{1}{9}\cos(3x+\pi) + C$$

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\cos(3x+\pi)\,dx = 2\left(\frac{1}{3}x\sin(3x+\pi) + \frac{1}{9}\cos(3x+\pi)\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\left(\left(\frac{1}{3}\cdot\frac{\pi}{3}\sin(\pi+\pi) + \frac{1}{9}\cos(\pi+\pi)\right) - \left(\frac{1}{3}\cdot\frac{\pi}{6}\sin\left(\frac{\pi}{2}+\pi\right) + \frac{1}{9}\cos\left(\frac{\pi}{2}+\pi\right)\right)\right)$$

Some physical applications of anti-derivative (& derivative)

$f(x)$: A physical quantity $\rightsquigarrow$ distance

anti derive — derive

$f'(x)$: rate of change in f $\rightsquigarrow$ velocity

anti derive — derive

$f''(x)$: rate of change of rate of change $\rightsquigarrow$ acceleration

<u>Example 1</u>: A runner's distance from the starting point is given by $S(t) = t^2$ t in seconds, S in meters.

How long does it take for the runner to hit a speed of 3 m/s ?

A. 6 sec   B. $2/3$ sec   C. $3/2$ sec   D. 3 sec

$$S(t) = t^2 \rightsquigarrow V(t) = 2t = 3 \Rightarrow t = 3/2 \text{ sec}$$

Example 2. A ball is thrown upward from a height of 5 m with a velocity of 4.9 m/s.

(a) At what moment the ball is at rest?

(b) When does the ball hit the ground?

(Assume the acceleration due to gravity is constant, that is $a(t) = -9.8$ m/s^2)

Next Class

Practice Problems :

a) $\int 3x e^{-x}\,dx$

b) $\int \frac{\ln x}{x^2}\,dx$

c) $\int x^2 \cos x\,dx$

d) $\int \ln x\,dx$

e) $\int e^{x} \sin x\,dx$

f) $\int x^3 e^{x^2}\,dx$

g) $\int \sin x \cos x\,dx$

h) $\int \cos x \cdot \ln(\sin x)\,dx$

Definite Integrals

a) $\int_{-1}^{1} (2x+8)^3(-x+2)\,dx$

b) $\int_{0}^{\pi/2} (-6x+4)\cos x\,dx$

c) $\int_{1}^{e^3} x^7 \ln x\,dx$

d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2x \cos(3x+\pi)\,dx$