Chapter 3. Systems of ODEs

Sept 26
Lecture 10

Many physical problems involve a number of separate but interconnected components such as:

(1) Separate masses in a mechanical system

(2) Species in a biological system

(3) Elements in a chemical system

(4) Voltage and current in an electrical system

In these cases, the mathematical model will be a system of two or more differential equations.

Chapter 1: 1^{st} order: only one equation

t: indep var

$x(t)$: dep var

$$\frac{dx}{dt} = f(t, x(t))$$

In a system: t: indep var

multiple dep. var. $x_1(t), x_2(t), \ldots, x_n(t)$

1^{st} order system of ODEs

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ dx_2/dt = f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ dx_n/dt = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

For example: $\begin{cases} x_1' = x_2 - x_1 - t \\ x_2' = x_1 + e^t \end{cases}$

In general: $\vec{X}'(t) = F(t, \vec{X}(t))$

where $\vec{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$: Solve the equation for $X(t)$.

Definition 1: A system of ODE is called autonomous if F does NOT depend explicitly on t (indep var

$$\vec{X}'(t) = F(\vec{X}(t))$$

Def 2: A system of ODE is called linear if it is in the form: (2×2 case)

$$\vec{X}'(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \vec{X}(t) + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

take

$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and matrix multiplication:

$$\begin{cases} x_1' = a(t)\cdot x_1 + b(t)\, x_2 + g_1(t) \\ x_2' = c(t)\, x_1 + d(t)\, x_2 + g_2(t) \end{cases}$$

Recall 1D case: linear ODE

$$y' + a(t)\, y + b(t) = 0$$

compare

<u>Def 3</u>: A <u>linear autonomous</u> system is of the form:

$$\vec{X}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{X}(t) + \underbrace{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}}_{\text{force term}}$$

where a, b, c, d, g_1, g_2 are numbers.

We restrict ourselves to homogeneous case (unforced case)

where $g_1 = g_2 = 0$:

$$\vec{X}'(t) = A\vec{X} \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Example:
$$\begin{cases} x_1'(t) = 3x_1(t) - 2x_2(t) \\ x_2'(t) = 2x_1(t) - 2x_2(t) \end{cases}$$

Rewrite:
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{x}' = A\vec{x} \qquad \text{linear, autonomous, hom.}$$

where $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

Recall 1D case: $y' = ay \xrightarrow{\text{separable}} y(t) = Ce^{at}$

Now we have

$$\underbrace{\vec{x}' = A\vec{x}}_{A \to \text{a matrix}} \longrightarrow \text{later: } e^{At} = ? \text{ meaning.}$$

* Since A is a constant matrix, we still expect to have "exponential nature" for the solution.

Let's guess a candidate for the solution is

$$\vec{x}(t) = \vec{v}\, e^{rt}$$

Let's find $\vec{v}$ and r ?

plug $x(t)$ into $x' = Ax$

$\vec{v}e^{rt}$ "

$$X' = \vec{v}\, r\, \cancel{e^{rt}} = A\,\vec{v}\, \cancel{e^{rt}} \leftarrow \boxed{\begin{array}{c} X' = AX \\ (\vec{v}e^{rt})' = A(\vec{v}e^{rt}) \end{array}}$$

$$\Rightarrow \boxed{r\vec{v} = A\vec{v}} \quad (\mathrm{I})$$

reminds you of ?

This is eigenvalue equation, we solve it for r and $\vec{v}$

Linear Algebra Review: Eigval and eigvec.

$$(\mathrm{II}) \quad (A - rI)\vec{v} = 0 \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For (II) to have a <u>non-zero</u> solution $\vec{v}$, we require

$$\det(A - rI) = 0$$

Go back to **Example**:

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$$

$* \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\det\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = (3-r)(-2-r) + 4 = 0$$

Characteristic equation of matrix A.

$$\Rightarrow \boxed{r^2 - r - 2 = 0}$$

$$\Rightarrow (r-2)(r+1) = 0 \quad \begin{array}{l} \rightarrow r = 2 \\ \rightarrow r = -1 \end{array}$$

eigvalues

Remark : $\boxed{\underbrace{\det(A - rI)} = 0}$ → Characteristic equation of A .

is a polynomial of degree n so

it can have n solutions so

n eigvalues including real, repeated and

imaginary eigvalues → (This example : two distinct real eigval's)

Go back to example ; find eigvec corresponding to each eigval:

$$r = 2 \Rightarrow (A - rI)\vec{v} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} v_1 - 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases} \longrightarrow \text{One equation } v_1 = 2v_2$$

$v_2 = 1 \rightsquigarrow v_1 = 2$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ → All other eigve are multiples of this .

* In general, there are infinitly many eigvec satisfying $v_1 = 2v_2$, we only choose one of them .

$$\boxed{\vec{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for } r_1 = 2}$$

$$r = -1 \Rightarrow \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} 4v_1 - 2v_2 = 0 \\ 2v_1 - v_2 = 0 \end{cases}$$

$$v_2 = 2v_1 \xrightarrow{v_1 = 1} \boxed{v_2 = 2 \rightsquigarrow \vec{v}^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

Eigvector matrix : $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

each column has an eigvec.

Solutions: $\quad X(t) = \vec{v}\, e^{rt}$

$$\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}$$

Question : What is general solution ? Check the facts on the next page.

Fact I $\Rightarrow$ $X = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

is also a solution.

Fact II and III $\Rightarrow$ Two solutions $\vec{x}_1$ and $\vec{x}_2$ are linearly independent

so

$$\boxed{X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}}$$

general solution (covers all possible solution)

Facts :

(1) Superposition Theorem :

If $\vec{x}^1$ and $\vec{x}^2$ are two solutions to $\vec{x}' = A\vec{x}$ then any linear combination of $\vec{x}^1$ and $\vec{x}^2$ is also a solution.

$$x = c_1 \vec{x}^1 + c_2 \vec{x}^2 \rightsquigarrow \text{linear Combination}$$

Verify : $x' = c_1 \vec{x}^{1\,\prime} + c_2 \vec{x}^{2\,\prime}$

$$= c_1 (A \vec{x}_1) + c_2 (A \vec{x}^2)$$

$$= A (\underbrace{c_1 \vec{x}_1 + c_2 \vec{x}^2}_{x})$$

$$= A \vec{x}$$

(2) If r_1 and r_2 are two distinct eigvalues and $\vec{v}_1$ and $\vec{v}_2$ their corresponding eigvec, then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent

if $c_1 \vec{v}_1 + c_2 \vec{v}_2 = 0$ then $c_1 = c_2 = 0$

(3) If $\vec{X}_1$ and $\vec{X}_2$ are two linearly indep. solutions of $\vec{X}' = A\vec{X}$ then all solutions of the system can be represented as

$$X(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$$ where

c_1 and c_2 are constants.