Last Class :

Lecture 2
Sept 7

DE : equation containing derivative, for example

ODE: $y(t)$

MATH 215/255

PDE: $y(t, x, z)$

$\frac{\partial y}{\partial t}$, $\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial z}$

$$\frac{dy}{dt} = y(t)$$

- Linear vs. nonlinear $\quad a(t)\frac{dy}{dt} + b(t)\,y(t) = c(t)$
  - * yy' NO
  - * y^2 NO
  - * e^{y} NO
  - * $\tan y$ NO
  - ⋮
- Order of an ODE : highest derivative

Today :

1. Separable Equations.

2. Integrating factor Method.

1. Separable Equation :

General Form : $$\frac{dy}{dx} = \overbrace{f(x)}^{\text{only function of } x} \cdot \underbrace{g(y)}_{\text{only function of } y.}$$

Goal ↓ Solve this for $y(x)$

How we solve :

We rewrite the equation to separate the two parts.

$\times\ dx$

$\div\ g(y)$ $\Longrightarrow$ $$\underbrace{\frac{dy}{g(y)}}_{\text{terms with } y} = \underbrace{f(x)\ dx}_{\text{terms with } x}$$

Integrate : $$\int \frac{dy}{g(y)} = \int f(x)\, dx + C$$

Use integration techniques to compute integrals.

Example 1. Solve $\begin{cases} \dfrac{dy}{dx} = \dfrac{x^2}{y} \\ y(0) = 1 \end{cases}$

$y' = \dfrac{x^2}{y}$ $\boxed{y'y = x^2}$

$\boxed{y' - \dfrac{x^2}{y} = 0}$ Non-linear

$$\frac{dy}{dx} = f(x) \cdot g(y) = \underbrace{x^2}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$$

Cross-multiply : $\quad y\,dy = x^2\,dx$

Integrate : $\quad \int y\,dy = \int x^2\,dx + C$

$+c_1$ $\quad +c_2$

* Each integral gives a constant, we make them into one constant C.

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

Implicit form : x and y combined, y has powers other than 1

Explicit form : $y = f(x)$

We solve for y to change to explicit :

* Note : Constant C might be different in each step, but we usually show them by C.

$$\xrightarrow[2C \rightsquigarrow C]{\times 2} \quad y^2 = \frac{2}{3}x^3 + C$$

$$y = \pm\sqrt{\frac{2}{3}x^3 + C}$$

Let's pick $\oplus$:

$$\boxed{y = \sqrt{\frac{2}{3}x^3 + C}}$$ General Solution

Use $y(0) = 1$ to find C :

$$1 = \sqrt{\frac{2}{3}\cdot 0 + C} \Rightarrow \sqrt{C} = 1 \Rightarrow \boxed{C = 1}$$

$$\boxed{y = \sqrt{\frac{2}{3}x^3 + 1}}$$ particular solution.

Observations : ⊛ We need to choose +/−

If we choose $y = -\sqrt{\frac{2}{3}x^3 + C}$

$$1 = -\sqrt{C} \Rightarrow \sqrt{C} = -1 \quad \times \text{ NOT possible}$$

⊛ If x is too negative then the solution fails to exist ⇝ under root becomes negative.

⇝ In most cases, the given situation will give us an idea of how to determine signs based on the context of the question.

Ex 2 : Solve $\begin{cases} y' = xy + x + y + 1 \\ y(0) = 2 \end{cases}$

Is it separable ?

Factor: $y' = x(y+1) + (y+1)$

$$\frac{dy}{dx} = y' = \underbrace{(y+1)}_{g(y)}\underbrace{(x+1)}_{f(x)}$$ Separable ✓

$$\boxed{y' + (x+1)y = x+1}$$ Linear.
$(a(x)y' + b(x)y = c(x))$

Separate:

$$\frac{dy}{y+1} = (x+1)\,dx$$

$$\int \frac{dy}{y+1} = \int (x+1)\,dx + C$$

or $\ln|y+1|$

$$\ln(y+1) = \frac{1}{2}x^2 + x + C \rightsquigarrow y = \cdots$$

take e: $$y+1 = \underbrace{e^{\frac{1}{2}x^2 + x + C}} = e^{\frac{1}{2}x^2+x} \cdot e^{C}$$

A new constant C

$$y + 1 = Ce^{\frac{1}{2}x^2 + x}$$

(this is always ⊕, so we can ignore | | for the ln integral)

$$y = C e^{\frac{1}{2}x^2 + x} - 1$$

$y(0) = 2 \quad \Rightarrow \quad 2 = C \underbrace{\cancel{e^{0}}}_{1} - 1$

$$\Rightarrow \quad \boxed{C = 3}$$

$$\Rightarrow \quad \boxed{y = 3 e^{\frac{1}{2}x^2 + x} - 1}$$

* Check that y is a solution.
plug it into the equation.

Practice Problem : Solve $\begin{cases} (t^2+1) - \dfrac{1}{x^2+1} \cdot \dfrac{dx}{dt} = 0 \\ x(0) = 0 \end{cases}$ in $(-\pi, \pi)$

Answer :

$$x(t) = \tan\left(t + \frac{t^3}{3}\right)$$

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(*) Always Start with checking for a separable eqt.

What if it's NOT separable ?!

Ex 3. Solve $\frac{dy}{dt} + \frac{2}{t} y = 3$ linear 1st order.

$$\frac{dy}{dt} = 3 - \frac{2}{t} y \to \frac{dy}{y} = \left(\frac{3}{y} - \frac{2}{t} \cancel{\frac{y}{y}}\right) dt$$

Can NOT make it separable.

Magic: Multiply through t^2 ⤳ integrating factor

$$t^2 \frac{dy}{dt} + 2t\, y = 3t^2$$

? Where did t^2 Come from ?!

We'll see soon ☺

Observe:

$$\frac{d}{dt}\left(t^2 y\right) = t^2 \frac{dy}{dt} + 2t \cdot y = \text{LHS}$$

product Rule

Do reverse product rule and rewrite:

$$\frac{d}{dt}\left(t^2 y\right) = 3t^2$$

integrate $\quad t^2 y = \int 3t^2\, dt + C$

* $\int \frac{d}{dt}(\text{whatever}) = \text{whatever}$

$$t^2 y = t^3 + C$$

$\div t^2$ $$\boxed{y = t + \frac{C}{t^2}}$$ general solution

2. Integrating Factor Method

↳ Always works for a <u>1st order linear</u> ODE

General form : $$a(x) \frac{dy}{dx} + b(x) y = c(x)$$

$p(x)$: known $g(x)$: known

$\div a(x)$ $$\frac{dy}{dx} + \frac{b(x)}{a(x)} y = \frac{c(x)}{a(x)}$$

$$\boxed{\frac{dy}{dx} + p(x) y = g(x)}$$ ★

<u>Goal</u>: <u>Find</u> a function $r(x)$ and multiply by $r(x)$ so that we make LHS into

$$\frac{d}{dx}(\text{something})$$ ⇝ reverse product rule can be applied then

Now, let's see how we find $r(x)$:

multiply $\star$ by $r(x)$

$$\underbrace{r(x)\frac{dy}{dx} + \underbrace{p(x)\,r(x)\,y}}_{\text{Make it } \frac{d}{dx}(\quad)} = g(x)\,r(x)$$

Compare

Observe :

$$\frac{d}{dx}\big(r(x)\,y(x)\big) = r(x)\frac{dy}{dx} + \underbrace{y\frac{dr}{dx}}$$

So we must have $\boxed{\dfrac{dr}{dx} = p(x)\,r(x)}$

Next class we solve , it'll give us $r(x)$

and we can solve $(\star)$ then.