Reminder :

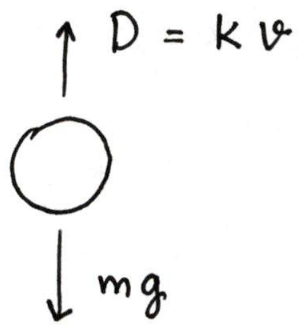
- Webwork 1 is due on Friday, Sept 14.
- Office hours :

  Thurs 4–5 pm
  Fri 4–5:30 pm

New location → $\boxed{\text{Location : MATH ANNEX Room 1118}}$

Last Class :

Falling Object Model :



$$a = \frac{dv}{dt}$$

N2L : $ma = F$

$$m\frac{dv}{dt} = mg - kv$$

1st order linear ODE ← $$\boxed{\frac{dv}{dt} = g - \frac{k}{m}v} \quad (\mathrm{I})$$

Use integrating factor method to solve (I)

Rewrite as

$$\frac{dv}{dt} + \underbrace{\frac{k}{m}}_{p(t)} v = \underbrace{g}_{g(t)}$$

integrating factor

$$r(t) = e^{\int p(t)\,dt} = e^{\int \frac{k}{m}\,dt} = e^{\frac{k}{m}t}$$

$$\frac{d}{dt}\left(e^{\frac{k}{m}t} v\right) = e^{\frac{k}{m}t} g$$

integrate and simplify

$$\boxed{v(t) = \frac{mg}{k} + C e^{-\frac{k}{m}t}}$$

general solution

* This is also a separable equation, rewrite (I) as

$$\frac{dv}{g - \frac{k}{m}v} = dt \quad \text{and integrate.}$$

$$\int \frac{dv}{g - \frac{k}{m}v} = t + C \Rightarrow -\frac{m}{k}\ln\left(g - \frac{k}{m}v\right) = t + C_1$$

$C_2 = e^{-\frac{kC_1}{m}}$

$$\Rightarrow g - \frac{k}{m}v = C_2 e^{-\frac{k}{m}t}$$

$C_3 = -\frac{m}{k}C_2$

$$\Rightarrow v = \frac{mg}{k} + C_3 e^{-\frac{k}{m}t}$$

Let's focus on the ODE itself and not on the solution.

$$\frac{dv}{dt} = g - \frac{k}{m} v \qquad (I)$$

For simplicity, assume $m = 10$ kg

$k = 2$

$g = 9.8\ m/s^2$

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \rightsquigarrow \text{solution: } v(t)$$

$\frac{dv}{dt} = v' =$ slope of the tangent line to $v(t)$.

Let's pick some values for v, and evaluate RHS:

$v = 40 \rightsquigarrow 9.8 - \frac{40}{5} = 9.8 - 8 = \underset{}{\overset{\frac{dv}{dt}}{1.8}} \rightsquigarrow$ slope of tangent line:

$v = 45 \rightsquigarrow \text{RHS} = 9.8 - 9 = 0.8 = \frac{dv}{dt}$

$v = 50 \rightsquigarrow \text{RHS} = 9.8 - 10 = -0.2 = \frac{dv}{dt}$

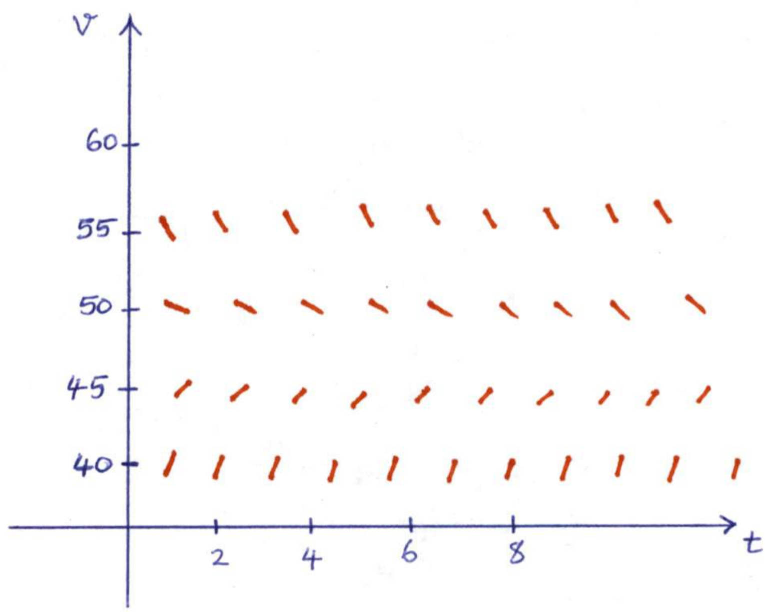
$v = 55 \rightsquigarrow \text{RHS} = 9.8 - 11 = -1.2 = \frac{dv}{dt}$

Corresponding to each v, we sketch a line with slope $= 1.8, 0.8, \ldots$ for all t.

* If we keep evaluating RHS and sketch the line segments corresponding to slopes we will get a $\boxed{\text{Slope field}}$ or $\boxed{\text{direction field}}$

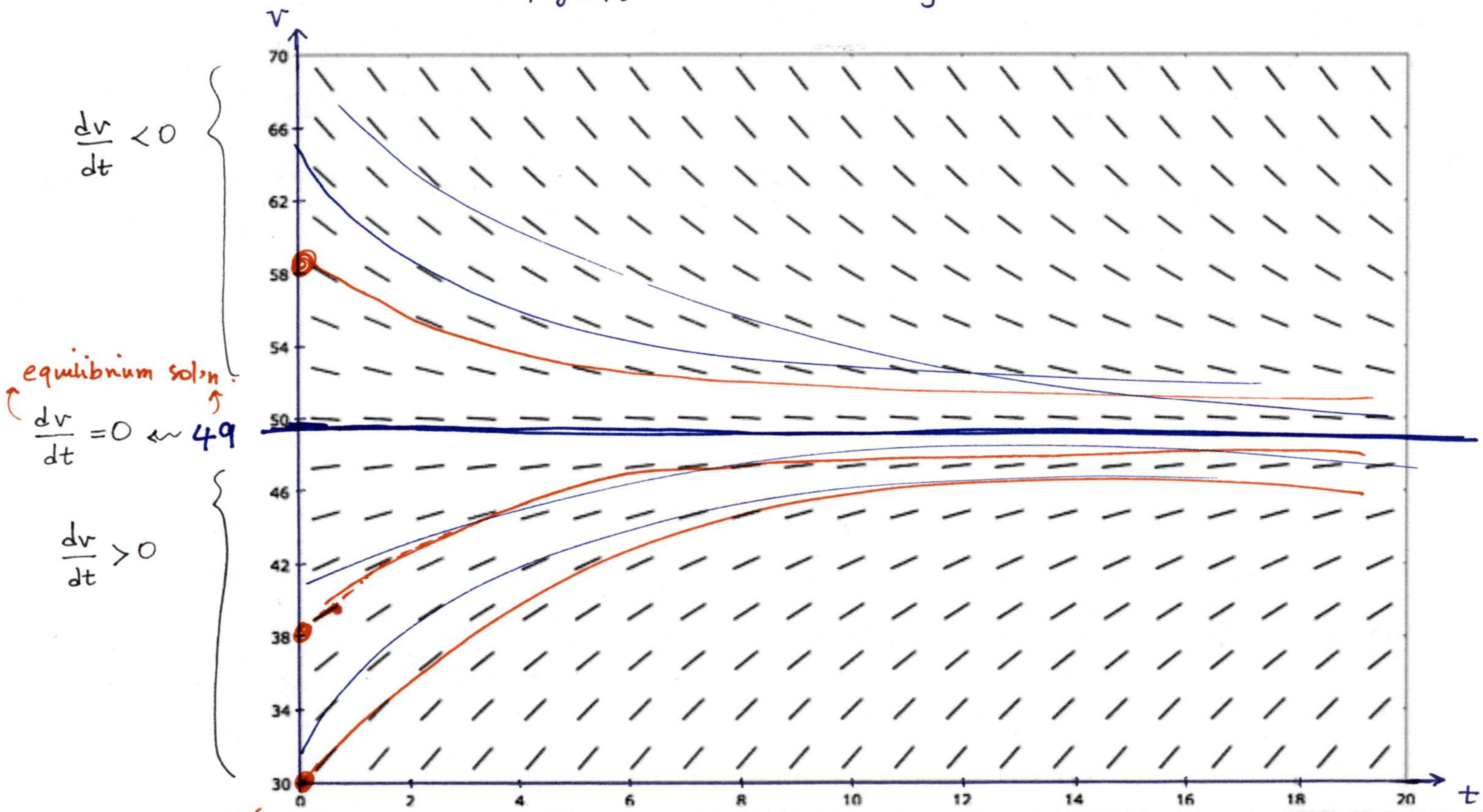
* MATLAB does this for us. Don't worry ☺

  (In a given interval for v and t, we make a grid fine enough so that it can represent the behaviour of the solution.)



Falling Object Model (Slope field)

$$\frac{dv}{dt} = g - \frac{k}{m} v \xrightarrow[k=2,\ g=9.8]{m=10\text{ kg}} \frac{dv}{dt} = 9.8 - \frac{v}{5}$$



This figure is a slope field or direction field.

Observations :

① Following the line segments starting from some initial value $v(0) = v_0$, we'll have a curve for the solution. (different initial value, different curve.)

② if v is below a certain critical value

then $\frac{dv}{dt} > 0$ ⇝ speed of the object is increasing as it falls.

if v is above the certain critical value

then $\frac{dv}{dt} < 0$ ⇝ Object slows down as it falls.

Question : What is the critical value?

$$\frac{dv}{dt} = 0$$

$$\Rightarrow 9.8 - \frac{v}{5} = 0 \Rightarrow \boxed{v = 49}$$

critical value

This critical value is called equilibrium solution.

$\star$ $v(t) = 49$ is a sol'n to DE (I).

Verify: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

$v(t) = 49 \rightsquigarrow \frac{dv}{dt} = 0 = 9.8 - \frac{49}{5} = 9.8 - 9.8$ ✓

→ Go back to the original equation:

$\star$ $\frac{dv}{dt} = g - \frac{k}{m} v$

equilibrium solution occurs when $\Rightarrow \frac{dv}{dt} = 0 = g - \frac{k}{m} v$

$$\Rightarrow \boxed{v = \frac{mg}{k}} \quad \text{equilibrium sol'n}$$

$$\Rightarrow \underbrace{kv}_{\text{drag}} = \underbrace{mg}_{\text{gravity}}$$

In the context of this question, equilibrium solution corresponds to perfect balance between gravity and drag.

(3) Checking the slopefield we see all the solution curves are converging to the equilibrium solution.

In this context:

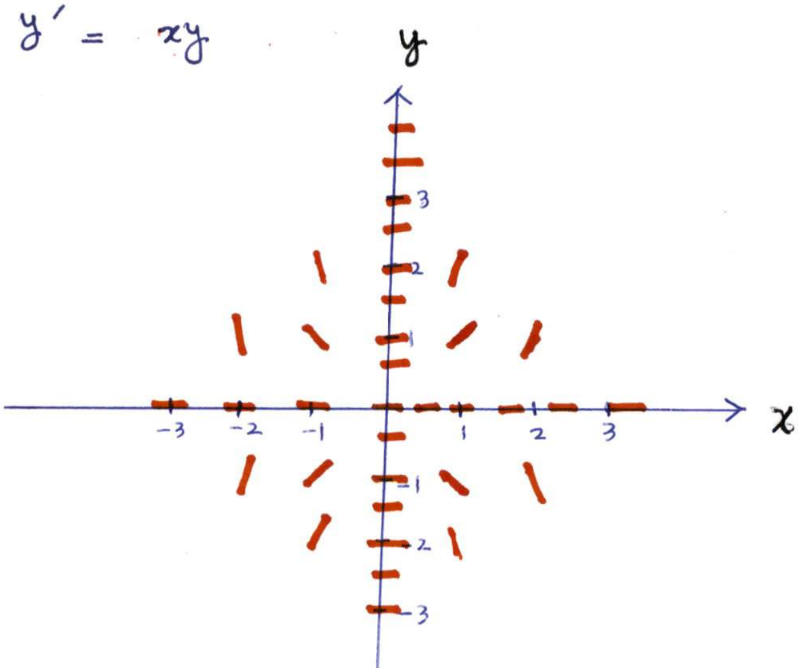
equilibrium solution = terminal velocity.

$$= \frac{mg}{K}$$

This can be verified from the solution as well:

$$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} \frac{mg}{k} + \underbrace{Ce^{-\frac{k}{m}t}}_{0} = \frac{mg}{K}$$

Example: $y' = xy$



$$\frac{dy}{dx} = xy$$

Since RHS depends on both x and y, we should evaluate at different x and y values.

$(0,0) \rightsquigarrow y' = 0$

$(0,y) \rightsquigarrow y' = 0 \rightsquigarrow$ on y-axis

$(x,0) \rightsquigarrow y' = 0 \rightsquigarrow$ on x-axis

$(1,1) \rightsquigarrow y' = 1 \leftsquigarrow (-1,-1)$

$(1,-1) \rightsquigarrow y' = -1 \leftsquigarrow (-1,1)$

Side Note: Compare RHS of

$$\frac{dy}{dx} = \underbrace{xy}_{\text{depends on } x \text{ and } y \text{ (dep. \& indep. var.)}} \quad \text{and}$$

$$\frac{dv}{dt} = \underbrace{g - \frac{k}{m}v}_{\text{only a function of } v \text{ (dependent var.)}}$$

* More about this in the next class.

Running this process for lots of points, the direction field looks like:

$$\frac{dy}{dx} = xy$$