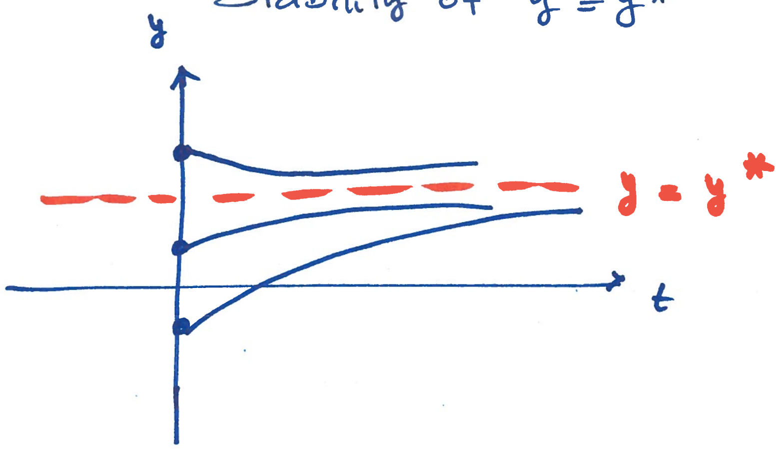
Sept 17
Lecture 6

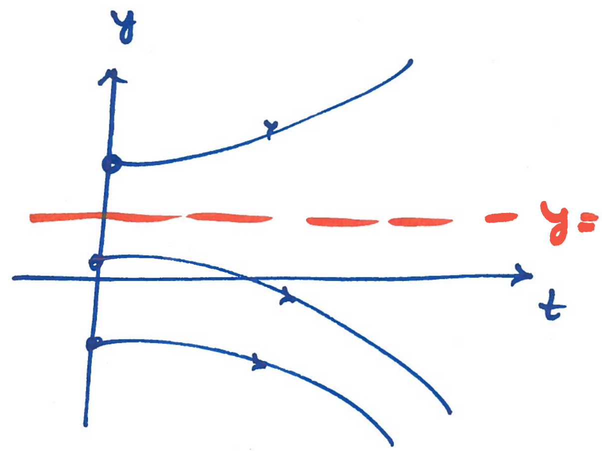
Last Class :

$$\frac{dy}{dt} = f(y) \quad \text{Autonomous eqt}$$

Equilib sol'n : $f(y) = 0 \rightsquigarrow \boxed{y = y^*}$

Stability of $y = y^*$





start close to y^*

$$\boxed{y(t) \longrightarrow y^* \text{ when } t \to \infty}$$

$y = y^*$ is Asymptotically stable

Asymptotically unstable.

Ex 1. $\dfrac{dy}{dt} = y^2 - y - 6$ Find steady states and classify.

$y^2 - y - 6 = 0 \Rightarrow (y-3)(y+2) = 0$

$\Rightarrow y = 3, \; y = -2$ steady states

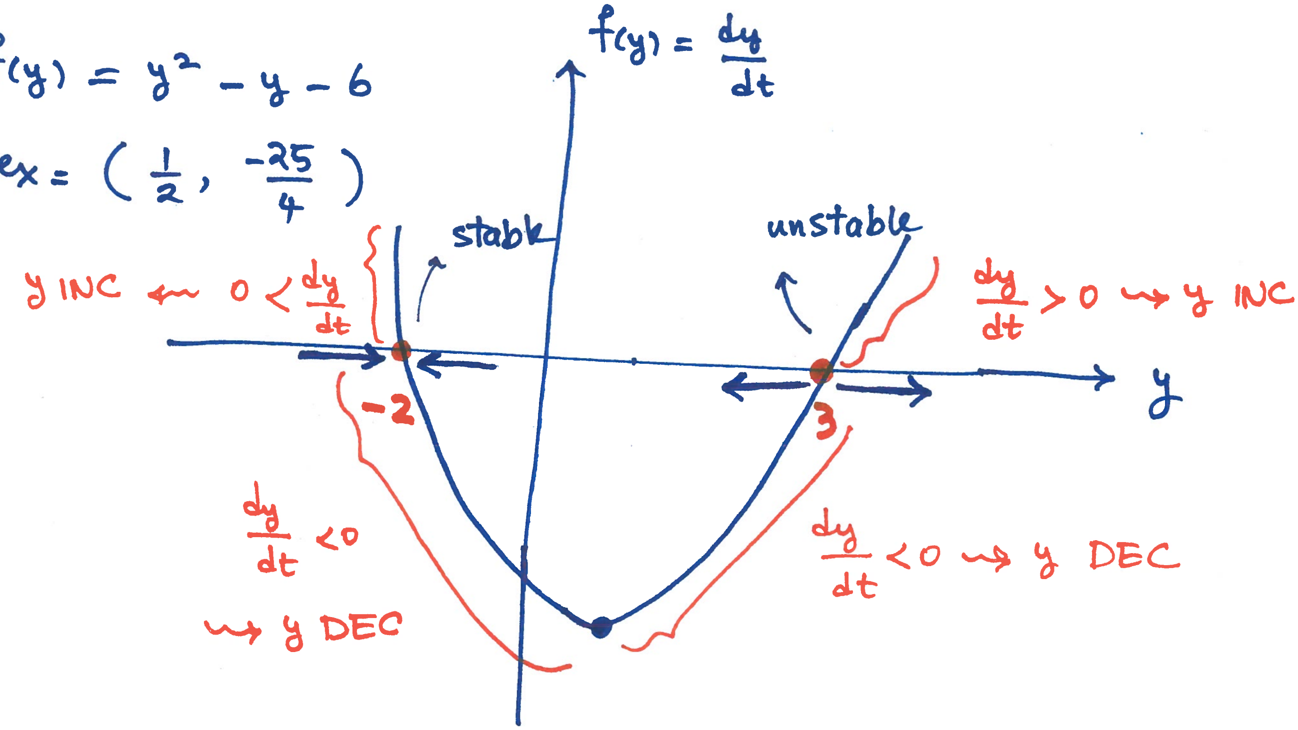
Asym. stable or unstable ?

$$\frac{dy}{dt} = y^2 - y - 6$$

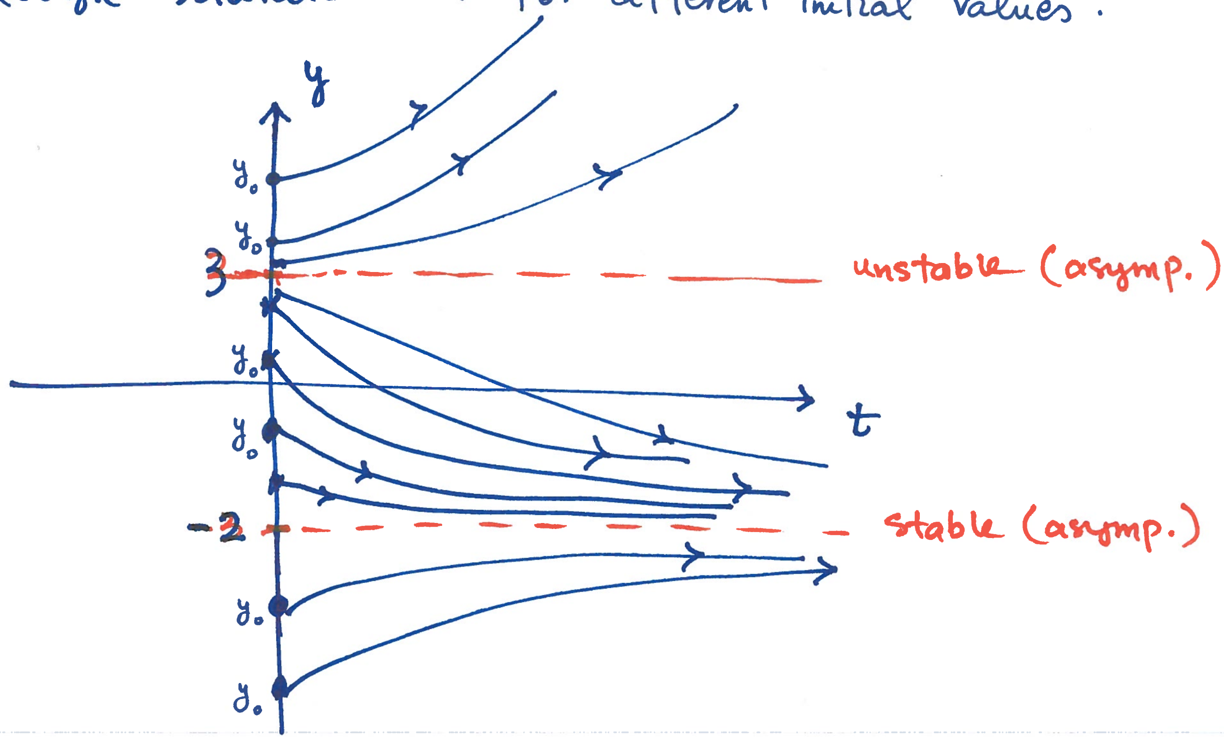
RHS is independent of t, so we can Sketch the graph $f(y) = \frac{dy}{dt}$ vs. y.

$f(y) = y^2 - y - 6$

Vertex $= \left(\frac{1}{2}, \frac{-25}{4}\right)$



a Rough solution curve for different initial values.



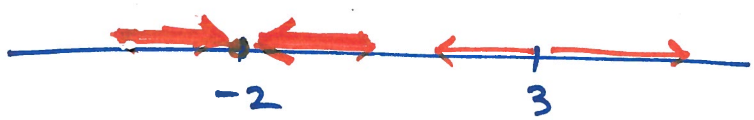
Shortcut: sign chart for $\frac{dy}{dt}$

y	test -3 ↑	-2	-1 ↑	3	test 4 ↑
$\frac{dy}{dt}$	$+$	0	$-$	0	$+$
y	→		←		→
		asym. stable		asym. unstable.	

phase line

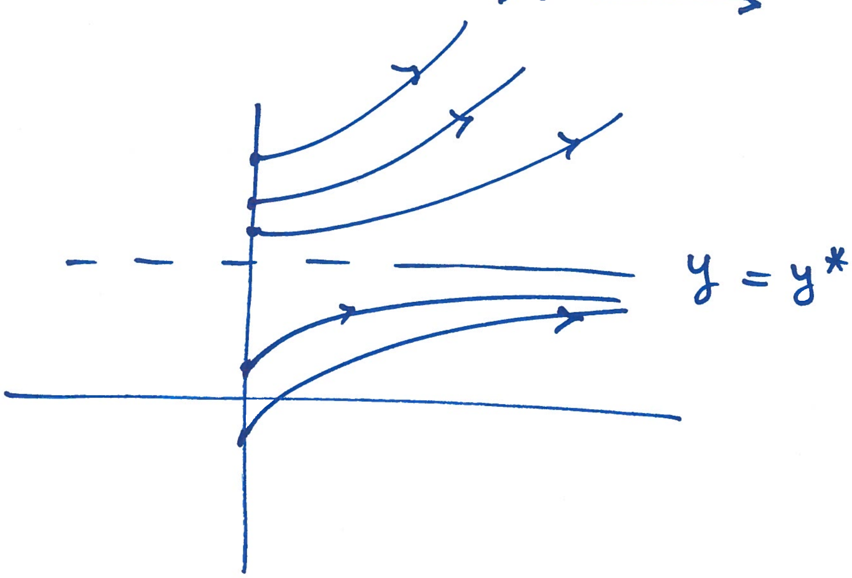
(phase portrait)

phase diagram



if phase line looks like

semi-stable steady state



Exercise. Classify steady states.

$$y' = (y^2 - 4)(y+1)^2$$

$$\frac{dy}{dt} = 0 \Rightarrow (y^2-4)(y+1)^2 = 0$$

$$\Rightarrow y = -2,\ 2,\ -1$$

Make a sign chart for $\frac{dy}{dt}$

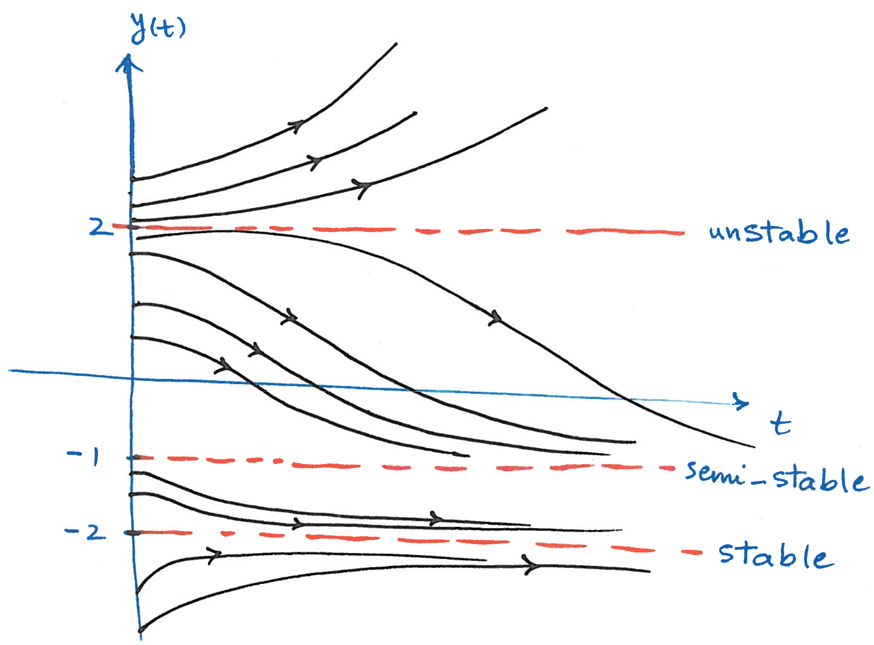
	test -3		test $-\frac{3}{2}$		test 0		test 3
y		-2		-1		2	
$\frac{dy}{dt}$	$+$	0	$-$	0	$-$	0	$+$
y	$\rightarrow$	$\bullet$	$\leftarrow$	$\bullet$	$\leftarrow$	$\bullet$	$\rightarrow$

Asym. stable $y = -2$

$y = -1$ semi-stable

$y = 2$ unstable

How do solution curves look like?



Existence and uniqueness of solution to 1^{st} order IVP

Initial-value problem

$$(*) \quad \begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

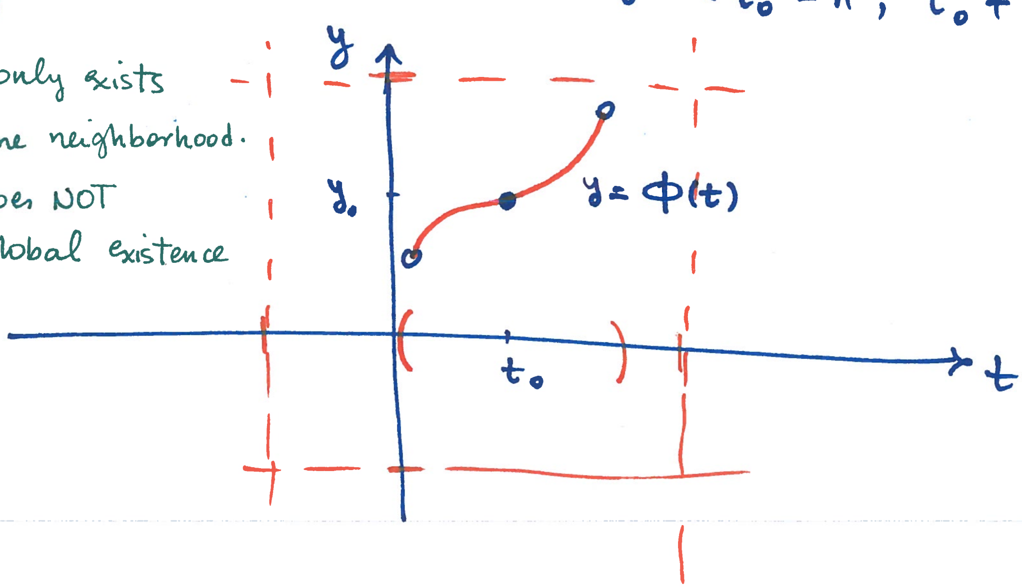
Question: Does $(*)$ always have a solution? NO

If so, is the solution unique? NO

Picard's Theorem:

If $f(t,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous in a rectangle neighborhood containing the point (t_0, y_0), then the IVP $(*)$ has a unique solution, $y = \phi(t)$, in some interval around t_0, say $(t_0 - h, t_0 + h)$.

The solution only exists locally in some neighborhood. The theorem does NOT guarantee global existence of solution.

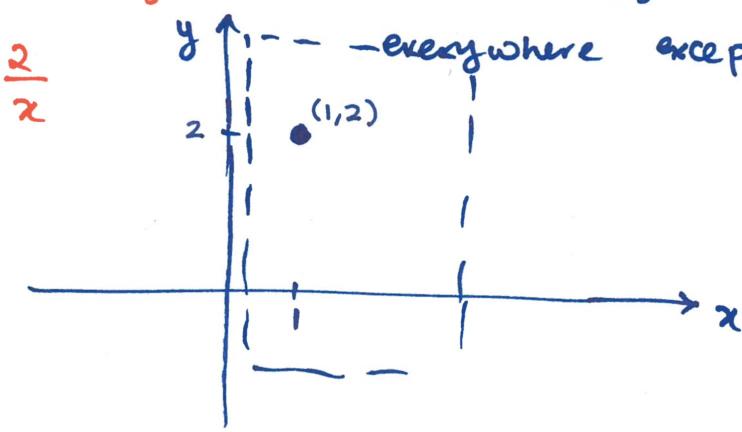


Ex 1 : Use Picard's Theorem to find an interval where the IVP
$$\begin{cases} x y' + 2y = 4x^2 \\ y(1) = 2 \end{cases}$$
has a unique solution.

$$\frac{dy}{dx} = \underbrace{-\frac{2}{x} y + 4x}_{f(x,y)}$$

f and $\frac{\partial f}{\partial y}$ are continuous everywhere except $x = 0$.

$$\frac{\partial f}{\partial y} = -\frac{2}{x}$$



Picard's Thm tells us that this IVP has a unique solution on any interval not containing 0.

In this example: $(0, \infty)$ is the interval of existence.

The initial value $\begin{cases} t_0 = 1 \\ y_0 = 2 \end{cases}$ imposes to choose $(0, \infty)$.

Apply integrating factor method $\rightsquigarrow$ $y(x) = x^2 + \frac{1}{x^2}$, $x > 0$

Remark. If IVP is 1^{st} order linear:

$$\begin{cases} \dfrac{dy}{dt} = -p(t)\,y + g(t) \\ y(t_0) = y_0 \end{cases}$$

$f(t,y) = -p(t)y + g(t)$

$\dfrac{\partial f}{\partial y} = -p(t)$ $\Rightarrow$ It is enough to have $p(t)$ and $g(t)$ continuous to apply Picard.

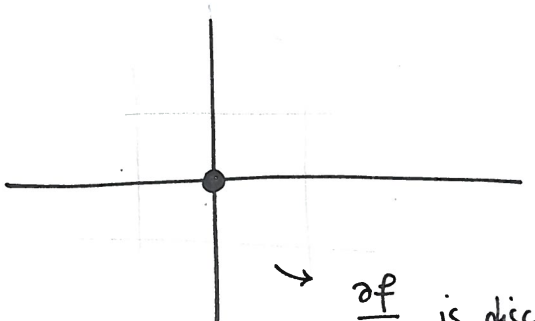
Ex 2:
$$\begin{cases} \dfrac{dy}{dt} = y^{\frac{1}{3}} & \text{for } t \geq 0 \\ y(0) = 0 \end{cases}$$

Does Picard's Thm apply?

$f(t,y) = y^{1/3}$ $\rightsquigarrow$ everywhere defined.

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-\frac{2}{3}} = \dfrac{1}{3} \dfrac{1}{\sqrt[3]{y^2}}$ $\rightsquigarrow$ discontinuous at $y = 0$

$\Rightarrow$ Picard's Thm does NOT guarantee existence and uniqueness of sol'n.



$\rightarrow$ $\dfrac{\partial f}{\partial y}$ is discontinuous at any neighborhood around the initial point $(0,0)$.

Solve : $\int y^{-\frac{1}{3}} dy = \int dt$

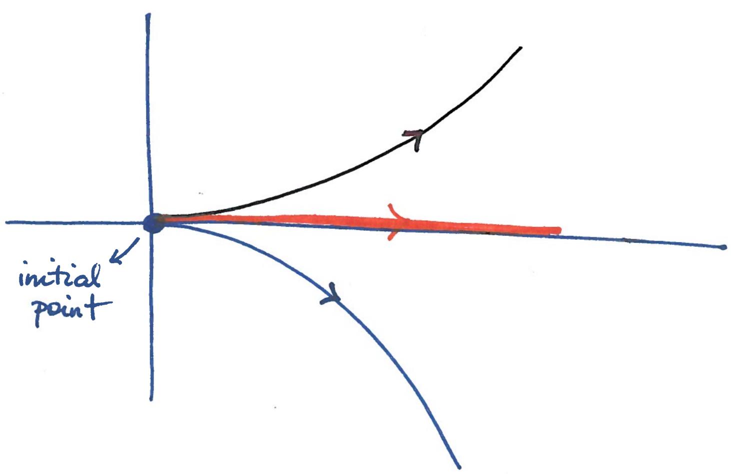
$\xrightarrow{y(0)=0}$ $\boxed{y(t) = \left(\frac{2}{3}t\right)^{3/2}}$

Also it has other solutions : $\boxed{\begin{array}{l} y(t) = -\left(\frac{2}{3}t\right)^{\frac{3}{2}} \\ \\ y(t) = 0 \end{array}}$

NO unique solution.

For one IVP $\begin{cases} \frac{dy}{dt} = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$, there are at least

3 solutions satisfying the same equation with $y(0) = 0$.

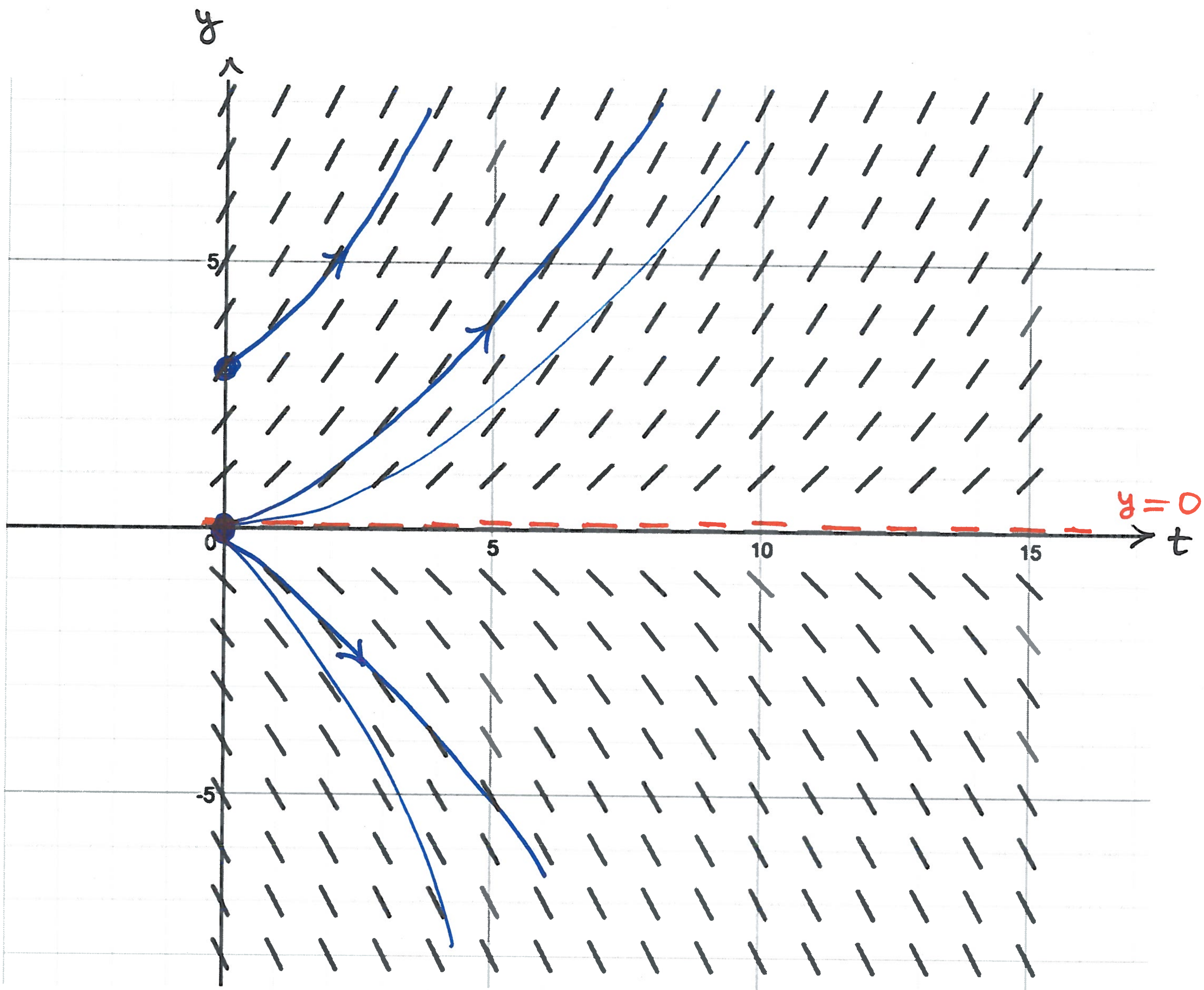


⇝ Check the slope-field

Note : When we say solution is unique, we mean the solution of the ODE with a given initial value. We know that when there's no initial value $y(t_0) = y_0$ given, we have a general solution and for different C values we have different curves. We are interested in having only one curve for a given $y(t_0) = y_0$

Slope field for $\frac{dy}{dt} = y^{\frac{1}{3}}$.

for $y(0)=0$, there's NO unique solution.



* Note: Change the initial value to $y(0)=1$ for example:

$f(y) = y^{\frac{1}{3}}$

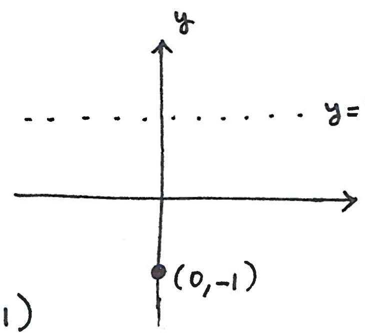
$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$ ⇝ can be continuous in a rectangle around $(0,1)$

⇒ For this initial value, Picard guarantees that there's an interval around 0, say $(0,\infty)$, for which IVP has a solution.

Practice Question. (a) Apply Picard's Theorem to the IVP

$$\frac{dy}{dt} = \frac{3t^2+4t+2}{2(y-1)} \quad , \; y(0) = -1$$

(b) Repeat this analysis when the initial condition is changed to $y(0) = 1$.



$$f(t,y) = \frac{3t^2+4t+2}{2(y-1)} \quad , \quad \frac{\partial f}{\partial y} = -\frac{3t^2+4t+2}{2(y-1)}$$

are Continuous everywhere except on the line $y=1$.

$\Rightarrow$ We can draw a rectangle around initial value $(0,-1)$ in which f and $\frac{\partial f}{\partial y}$ are cont's and Picard's Thm guarantees that this IVP has a unique solution is some interval around $t_0 = 0$.

Note : This does NOT necessarily mean that the solution exists for all t .

(b) if $y(0)=1$, then no rectangle can be drawn around $(0,1)$ such that f and $\frac{\partial f}{\partial y}$ be Cont's $\Rightarrow$ Picard's Thm does NOT say anything about possible solutions .

Let's separate the variables and solve :

$$2\int (y-1)\, dy = \int (3t^2+4t+2)\, dt + C \Rightarrow y^2 - 2y = t^3 + 2t^2 + 2t + C$$

$$\xrightarrow[\text{square}]{\text{Complete the}} y = 1 \pm \sqrt{t^3+2t^2+2t+C+1}$$

if $y(0) = -1 \Rightarrow \pm\sqrt{C+1} = -2 \xrightarrow[\text{is OK}]{\text{only } \ominus} C=3 \Rightarrow y = 1 - \sqrt{t^3+2t^2+2t+4}$

But if $y(0) = 1 \Rightarrow C = -1 \xrightarrow[\text{OK}]{\text{Both } \pm} y = 1 \pm \sqrt{t^3+2t^2+2t} \Rightarrow$ two solution

$\Rightarrow$ Uniqueness fails .

Ex. Solve $\begin{cases} y' = y^2 \\ y(0) = A \end{cases}$ and argue about

existence of solution for different values of A.

$$\frac{dy}{y^2} = dt \rightsquigarrow y = -\frac{1}{t+C}$$

$y(0) = A$

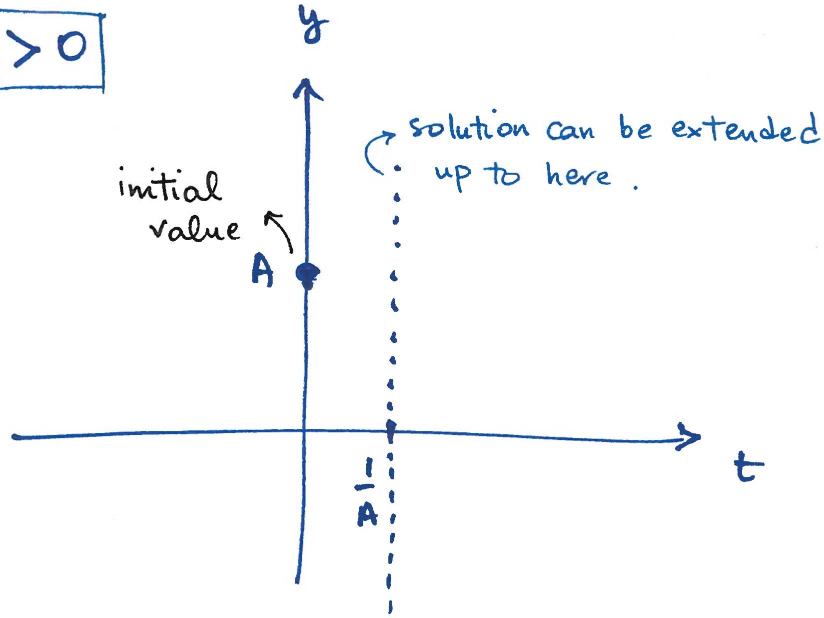
$\Rightarrow C = -\frac{1}{A}$ $\rightsquigarrow$ $y(t) = -\dfrac{1}{t - \frac{1}{A}}$

Solution blows up when $t \to \frac{1}{A}$.

$(y(t) \to \infty)$

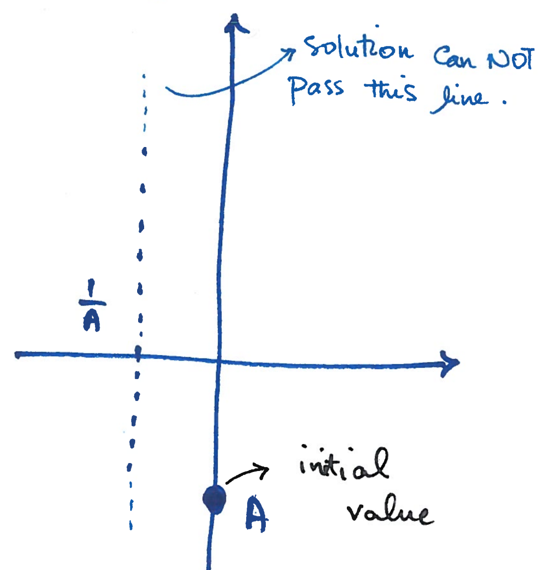
* This is a nice IVP, $f = y^2$ and $\frac{\partial f}{\partial y} = 2y$ everywhere defined and cont's but the solution does NOT exist everywhere.

$\boxed{A > 0}$



Interval of existence : $\left(-\infty, \frac{1}{A}\right)$

$\boxed{A < 0}$



$\left(\frac{1}{A}, \infty\right)$

$A = 0$

$y(0) = 0 \rightsquigarrow y(t) = 0$ Solution.

Note that when $A = 0$ the separation of variables won't work.

$$\frac{dy}{y^2} = dt \quad \text{when } y \neq 0$$

Check the slope field.

$$\begin{cases} \dfrac{dy}{dt} = y^2 \\ y(0) = A \end{cases}$$

$\rightsquigarrow$ See in the slope field :

for $A > 0$, solution $y(t) \longrightarrow \infty$. when $t \rightarrow \frac{1}{A} > 0$

for $A < 0$, solution $y(t) \longrightarrow -\infty$ when $t \rightarrow \frac{1}{A} < 0$