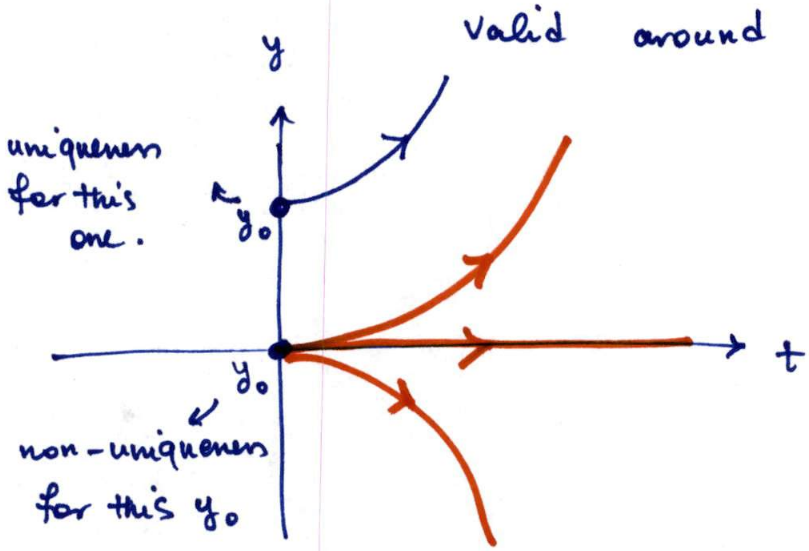
Last Class.

Picard's Thm $\rightarrow$ Existence and Uniqueness of sol'n to a 1^{st} order ODE with given initial value

$$\text{IVP} \quad \begin{cases} \dfrac{dy}{dt} = f(t,y) \\ \boxed{y(t_0) = y_0} \end{cases}$$

$\hookrightarrow$ Condition and Conclusion of the Thm is valid around (t_0, y_0)



$\Rightarrow$ Everything is local around initial value (t_0, y_0).

You change the initial value $\rightsquigarrow$ Conclusion may change

So far our toolbox: of 1^{st} order ODE

1) Separable equation

2) 1^{st} order linear ODE

↳ integrating factor

today ⇝ 3) Exact Equations

Calculus Review:

Partial Derivatives:

$$y = f(x) \rightsquigarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f: function of 1 variable

$$f(x,y) \longrightarrow \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$\frac{\partial f}{\partial x} \to f_x$

f: function of 2 variables

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$f_y \leftarrow \frac{\partial f}{\partial y}$

Ex: $f(x,y) = x^3 + x\,y^2_{(x)}$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2$$

$$\frac{\partial f}{\partial y} = x \cdot 2y = 2xy$$

$\xrightarrow[\text{total derivative}]{\text{if } y = y(x)}$ $df = 3x^2 + 1 \cdot y^2 + x \cdot \underbrace{2y\frac{dy}{dx}}_{\text{Chain rule}}$

df or $\frac{df}{dx}$

implicit differentiation

$$df = \boxed{3x^2 + y^2} + \boxed{2xy}\frac{dy}{dx}$$

$f(x,y)$: a function of two variables but $y = y(x)$.

$\Rightarrow f(x, y(x))$

then total derivative

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}}$$

Ex. $f(x,y) = x^3 e^{y(x)}$

$$\frac{\partial f}{\partial x} = 3x^2 \cdot e^y$$

$$\frac{\partial f}{\partial y} = x^3 e^y$$

$$\xrightarrow{y = y(x)} \frac{df}{dx} = 3x^2 e^y + x^3 e^y \cdot \frac{dy}{dx}$$

<u>Mixed partial derivatives</u> :

$$f(x,y) = x^3 + xy^2$$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 \longrightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2y$$

$$\frac{\partial f}{\partial y} = 2xy \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2y$$

=

Remark: If f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable

(Equality of mixed partials) then

$$\frac{\partial}{\partial y}\Big(\overbrace{\frac{\partial f}{\partial x}}^{f_x}\Big) = \frac{\partial}{\partial x}\Big(\overbrace{\frac{\partial f}{\partial y}}^{f_y}\Big)$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$$

$$(f_x)_y = (f_y)_x \quad \text{or} \quad f_{xy} = f_{yx}$$

Exact Equations: $\rightsquigarrow$ usually in physics

$\hookrightarrow$ Conservation law (conservation of energy)

Consider a 1^{st} order ODE of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (\mathrm{I})$$

or

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Suppose there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \xrightarrow[(\mathrm{I})]{\text{Rewrite}} \underbrace{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}}_{dF} = 0$$

$$\frac{\partial F}{\partial y} = N(x,y) \qquad \Rightarrow dF = 0$$

$$\Rightarrow F(x,y) = \underbrace{C}_{\text{constant}}$$

$\Rightarrow F(x,y) = C$ is a solution to (I)

($\exists$)

<u>Definition</u>. We call the ODE (I) exact if there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad , \quad \frac{\partial F}{\partial y} = N$$

$$\hookrightarrow \quad \text{LHS of I} = dF = 0$$

potential Function.

How do we know an ODE is exact ?

$$M + N \frac{dy}{dx} = 0$$

If exact $\rightsquigarrow$ $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$ $\rightsquigarrow$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial M}{\partial y}$$

$$=$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}$$

$$=$$

In order to have an exact equation:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{or} \quad M_y = N_x}$$

* This result is true both ways

Equation is exact $\Longleftrightarrow M_y = N_x$

How to find F ?

Ex. $\begin{cases} \dfrac{dy}{dx} = \dfrac{-2x-y}{x-1} \\ y(0)=1 \end{cases}$ $\Rightarrow$ $\overbrace{(x-1)}^{N}\dfrac{dy}{dx} + \overbrace{(2x+y)}^{M} = 0$

Exact? $M_y = 1$ ✓ Yes!
$N_x = 1$

So $\exists$ $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M = 2x+y$$

$$\frac{\partial F}{\partial y} = N = x-1$$

and $F(x,y) = C$ is a solution, now we find F:

$$\frac{\partial F}{\partial x} = 2x+y \xrightarrow[\text{w.r.t } x]{\text{integrate}} F(x,y) = x^2 + xy + g(y) + \cancel{C}$$

$$\xrightarrow{\frac{\partial F}{\partial y} = N} \frac{\partial F}{\partial y} = x + \underbrace{g'(y)} = x \underbrace{-1}$$

$$\Rightarrow g'(y) = -1 \Rightarrow g(y) = -y + \cancel{C}$$

$$\Rightarrow F(x,y) = x^2 + xy - y + \cancel{C}$$

Solution:

$$F(x,y) = C \Rightarrow x^2 + xy - y = C$$

$$\xrightarrow{y(0)=1} \quad -1 = C$$

$$\Rightarrow \quad x^2 + xy - y = -1$$ → implicit sol'n

Solve for y

$$\Rightarrow \quad y(x-1) = -1 - x^2$$

$$\Rightarrow \quad \boxed{y = \frac{1+x^2}{1-x}}$$ explicit. Solution

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Non-exact equations:

$$\text{(I)} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{non-exact}$$

Sometimes we can convert it to exact.

How? multiply (I) by a function $r(x,y)$

$$\text{(II)} \quad rM + rN\frac{dy}{dx} = 0$$

such that (II) becomes exact, i.e.

$$\frac{\partial(rM)}{\partial y} = \frac{\partial(rN)}{\partial x}$$