HW 1 is due tonight.

WebWork 2 is open.

Lecture 8
Sept 21

Last class:

Exact Equations:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (\mathrm{I})$$

(I) is exact if $\exists\, F(x,y)$ such that:

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \;\Rightarrow\; \text{LHS} = dF = 0$$

$$\Rightarrow \boxed{F(x,y) = C}$$

↳ solution to (I)

Condition for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (M_y = N_x)$$

- If (I) is NOT exact, it's possible to convert it to exact by finding a function $r(x,y)$ such that $$rM + rN\frac{dy}{dx} = 0 \quad (\mathrm{II})$$ becomes exact.

For (II) to be exact:

$$(rM)_y = (rN)_x$$

$$\xrightarrow[\text{rule}]{\text{product}} r_y M + r M_y = r_x N + r N_x$$

$$r(M_y - N_x) = r_x N - r_y M \qquad \text{(III)}$$

Solve III for $r(x,y)$ and II becomes exact.

$r(x,y)$ is called the <u>integrating factor</u>.

<u>Exercise</u>: Show that if (I) is a 1^{st} order <u>linear</u> ODE i.e.

$$\frac{dy}{dx} + p(x)\, y - g(x) = 0$$

then $r(x) = e^{\int p(x)\,dx}$ is the integrating factor that makes the ODE exact.

* <u>Remark</u>: Equation III might have more than one solution, so integrating factor need not to be unique.

Equation III NOT easy to solve in general so consider two special cases:

$r(x,y) = r(x)$

then $r_y = 0$ partial derivative becomes ordinary derivative

and $\frac{\partial r}{\partial x} = r_x = \frac{dr}{dx}$

then III becomes

$$\frac{dr}{r} = \frac{M_y - N_x}{N}\,dx$$

a function of x

a function of x

$$\boxed{r(x) = e^{\int \frac{M_y - N_x}{N}\,dx}}$$

integrating factor

$r(x,y) = r(y)$

so $r_x = 0$

and III becomes

$$\frac{dr}{r} = -\frac{M_y - N_x}{M}\,dy$$

a function of y only.

Similar

$$r(y) = e^{-\int \frac{M_y - N_x}{M}\,dy}$$

multiply (I) by r and solve the resulting exact equation. (II)

Summary:

Non-exact equations $\xrightarrow{\text{search}}$ for r

$\frac{M_y - N_x}{N} \rightsquigarrow$ function of x $\Rightarrow r(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$

$\frac{M_y - N_x}{M} \rightsquigarrow$ function of y $\Rightarrow r(y) = e^{-\int \frac{M_y - N_x}{M}\,dy}$

Multiply Equation by r and solve by exact equations method.

$\underline{\underline{\text{Ex}}}$. Solve $\quad \underbrace{(3xy+y^2)}_{M} + \underbrace{(x^2+xy)}_{N}y' = 0$

$$\left.\begin{array}{l} M_y = 3x+2y \\ N_x = 2x+y \end{array}\right\} \neq \rightsquigarrow \text{NOT exact}$$

Check for integrating factor:

$$M_y - N_x = 3x+2y-2x-y$$
$$= x+y$$

$$\frac{M_y - N_x}{N} = \frac{x+y}{x(x+y)} = \frac{1}{x} \rightsquigarrow \text{function of } x$$
$$\rightsquigarrow r(x,y) = r(x)$$

$$r(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x \longrightarrow \text{integrating factor}$$

$$\xrightarrow{\times x} \quad (3x^2y + xy^2) + (x^3 + x^2y)\,y' = 0$$

verify: this is exact now.

$\exists\ F(x,y)$ s.t.

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2 \xrightarrow[\text{over } x]{\text{integrate}}$$

$$\xrightarrow{F=?}$$

$$\frac{\partial F}{\partial y} = x^3 + x^2y$$

$$F(x,y) = x^3 y + \frac{1}{2}x^2 y^2 + g(y) + C$$

$$\xrightarrow[\text{derive w.r.t } y]{\frac{\partial F}{\partial y} = N} \quad x^3 + x^2 y + g'(y) = x^3 + x^2 y$$

$$\Rightarrow \quad g'(y) = 0 \quad \Rightarrow g(y) = C_1$$

So $F(x,y) = x^3 y + \frac{1}{2}x^2 y^2 + C_1$

Solution $F(x,y) = C$

$$\boxed{x^3 y + \frac{1}{2}x^2 y^2 = C}$$

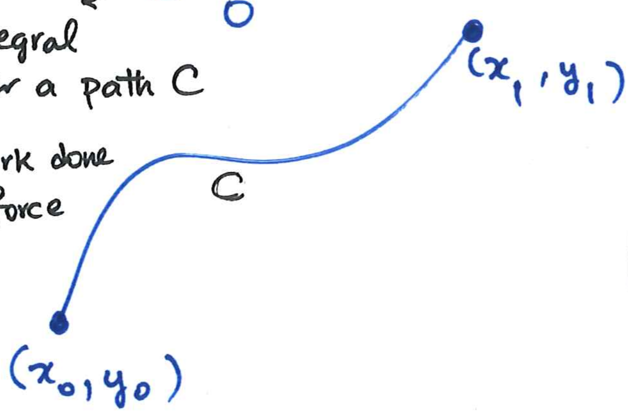
Physical interpretation of exact equations

$$W = \int_C dF \cdot dr = \int_C 0 = F(x_1,y_1) - F(x_0,y_0)$$

$\underset{\text{exact}}{=} 0$

integral over a path C

work done by force Φ



* The work done by force Φ only depends on the endpoints of the path.

* If $\vec{\Phi} = M\vec{i} + N\vec{j}$ is a vector field

$$\xrightarrow[\text{calculus}]{\text{vector}} \quad \int \Phi \cdot dr = \int (M,N)\cdot(dx,dy)$$

$$= \int M\,dx + N\,dy$$

exact equation: $\exists F$ $= \int_C dF = 0$

$$= F(x_1,y_1) - F(x_0,y_0)$$

Numerical methods to approximate the solution.

Euler's Method:

General idea: link (tangent) line segments and
not really !!!
construct a piecewise linear function that approximates the solution.

Consider IVP $\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ → slope of tangent line to the curve of the solution at different (t,y).

Procedure:

Start at (t_0, y_0) and find the tangent line at (t_0, y_0)

$$m_{tan} = \left.\frac{dy}{dt}\right|_{(t_0,y_0)} = f(t_0, y_0)$$

$$y = y_0 + f(t_0, y_0)\cdot(t - t_0)$$

↳ pick this line on a short interval (t_0, t_1).

Repeat the process with (t_1, y_1)

Write the equation of the line through (t_1, y_1)

with $$m = \left.\frac{dy}{dt}\right|_{(t_1,y_1)} = f(t_1, y_1)$$

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

<u>Next Class</u> :