Midterm Review

Chapter 1 : 1^{st} order ODE : $\frac{dy}{dt} = f(t,y)$ $(\star)$ $\left(\text{or } \frac{dy}{dx} = f(x,y) \right)$

3 methods to solve:

(1) Separation of variables when $(\star)$ is separable

(2) Integrating factor when $(\star)$ is 1^{st} order linear

(3) When $(\star)$ is exact $\rightarrow$ also integrating factor.

1. Separable equation:

Form : $\frac{dy}{dt} = f(t)\,g(y)$ or $\frac{f(t)}{g(y)}$ or $\frac{g(y)}{f(t)}$

Method : Separate each variable at one side of the equation & integrate.

2. Linear (1^{st} order) ODE :

Form : $\frac{dy}{dt} + P(t)\,y = g(t)$

Method : Integrating Factor : $r(t) = e^{\int P(t)\,dt}$

multiply both sides by $r(t)$ and use reverse product rule on LHS to have

$$\frac{d}{dt}\left(e^{\int P(t)\,dt} \cdot y \right) = e^{\int P(t)\,dt} \cdot g(t)$$

$$\Rightarrow \quad y(t) = e^{-\int P(t)\,dt}\left(\int g(t) \cdot e^{\int P(t)\,dt}\,dt + C \right)$$

$\circledast$ Things to Remember :

- $P(t)$ is on LHS in the standard form. Make sure you have the correct sign for $P(t)$ when finding the integrating factor
- Do NOT forget C : the constant of integration as you will lose one part of the solution without C.

3. Exact equations.

Form: $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

(or translate everything to $M(t,y) + N(t,y)\dfrac{dy}{dt} = 0$)

Method: (I) check for exactness

$M_y \stackrel{?}{=} N_x$

if Yes: (II) Find a function $F(x,y)$ such that $F_x = M$ and $F_y = N$

then $F(x,y) = C$ is a solution

if NO: (II) Find an integrating factor that makes the equation exact:

- $\dfrac{M_y - N_x}{N}$ a function of $x \rightsquigarrow r(x) = e^{\int \frac{M_y - N_x}{N} dx}$

- $-\dfrac{M_y - N_x}{M}$ a function of $y \rightsquigarrow r(y) = e^{-\int \frac{M_y - N_x}{M} dy}$

Multiply the equation by $r(x)$ (or $r(y)$) it should become exact and then find $F(x,y) = C$.

How to find $F(x,y) = ?$

$F_x = M \rightsquigarrow$ integrate with respect to x

$F_y = N$ $\rightsquigarrow F(x,y) = \int M(x,y)\,dx + g(y)$

$\rightsquigarrow$ differentiate w.r.t y and equate to N to find $g'(y)$

$\rightsquigarrow$ find $g(y)$ using $g'(y)$.

* If it's easier, start with integrating $F_y = N$ over y and do similarly

$F(x,y) = \int N(x,y)\,dy + h(x) \underset{\text{differentiate}}{\xrightarrow{h'(x)}}$ find $h(x)$.

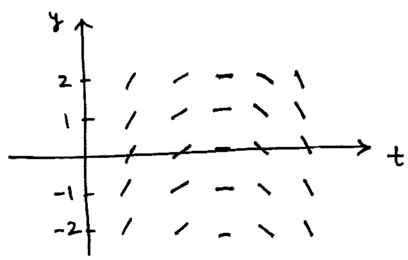
(*) Things to remember in all 3 methods :

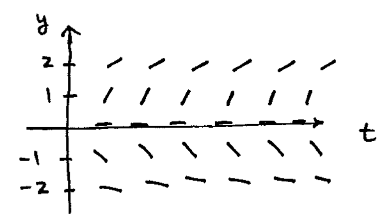
- Given the initial condition $y(t_0) = y_0$, you should find C.
- You should know integration techniques : substitution, integ. by parts, ...
- If the question states "solve for y", you must have your solution explicitly i.e. $y = f(x)$ or $y = f(t)$. Do NOT forget to convert implicit form to explicit by solving for y (especially in separable & exact equations.)

Slope field / direction field : $\frac{dy}{dt} = f(t,y)$ $(\star)$

- Evaluate $f(t,y)$ at different (t,y) values gives slope of the tangent line at each (t,y) which are represented by small line segments.

- If $(\star)$ is <u>autonomous</u> : $\frac{dy}{dt} = f(y)$ then slopes do not change with time so for each y, you'll have equal slopes at all t :





- If $(\star)$ is $\frac{dy}{dt} = f(t)$, then slopes do not change with y, so at each time equal slopes for all y.

<u>Interesting Case</u> : Autonomous

$\frac{dy}{dt} = f(y)$ $\rightsquigarrow$ y-values for which $f(y) = 0$ are called critical point or equilibrium solutions. (zero slope)

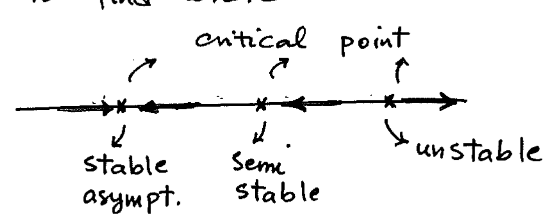
- Stability of Equilib. Solutions

* Determine the sign of $f(y) = \frac{dy}{dt}$ to find where

$\frac{dy}{dt} > 0 \rightsquigarrow y$ increasing

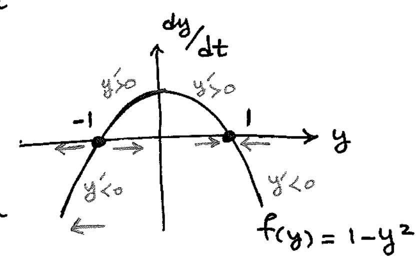
$\frac{dy}{dt} < 0 \rightsquigarrow y$ decreasing

$\xrightarrow[\text{line}]{\text{phase}}$



* You may be asked to sketch the graph of $\frac{dy}{dt}$ vs. y and make some observations from that graph

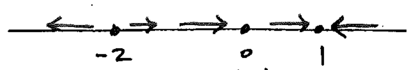
For example: $\frac{dy}{dt} = -(y-1)(y+1)$



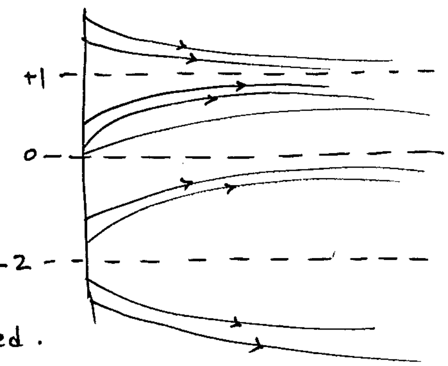
You should be able to translate the phase diagram to the graph or vice versa.

* You should also be able to sketch the trajectories of the solution using the behaviour observed in the phase line.

For example: $\frac{dy}{dt} = (1-y)(2+y)y^2 \rightsquigarrow$ critical points $y = 1, -2, 0$



unstable semi stable asym. stable



Conditions on $y(t_0) = y_0$

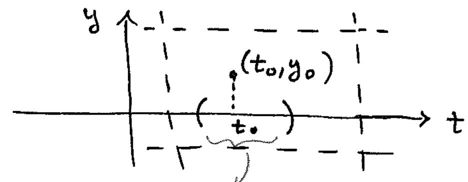
if $y_0 > 1$ or $0 < y_0 < 1 \rightsquigarrow$ solutions converge to 1.

if $-2 < y_0 < 0 \rightsquigarrow$ solutions converge to 0.

if $y_0 < -2 \rightsquigarrow$ solutions blow up, become unbounded.

Picard's Theorem (Existence & Uniqueness of a solution)

Let $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$ be an IVP. If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle around the initial point (t_0, y_0) then there exists a unique solution on some interval around t_0



There is a unique solution only on a small interval around t_0.

Important points :

- Picard's Thm only makes sense with the presence of an initial condition. Change (t_0, y_0) conclusion may be different
- Picard's Thm only guarantees existence of a unique solution in some small interval around t_0 and in general we can NOT extend this interval except when the equation is linear :

$$\frac{dy}{dt} + p(t)\, y = g(t) \quad \text{then if } \overbrace{p(t) \text{ and } g(t)}^{f \text{ and } \frac{\partial f}{\partial y}} \text{ are continuous on}$$

some interval I, then a unique solution exists for all t in I.

→ Compare the two cases below:

(1) $\begin{cases} \frac{dy}{dt} = t\,y \quad (\text{linear}) \\ y(0) = A \end{cases}$

$\begin{cases} f(t,y) = ty \\ \frac{\partial f}{\partial y} = t \end{cases} \Rightarrow$ Continuous everywhere around $(0, \ )$

$\Rightarrow I = (-\infty, \infty)$

$\overset{\text{Picard}}{\Rightarrow}$ A unique solution exists for all t on $(-\infty, \infty)$.

(No need to solve and verify.)

(2) $\begin{cases} \frac{dy}{dt} = t\,y^2 \quad (\text{non-linear}) \\ y(0) = A \end{cases}$

$\begin{cases} f(t,y) = ty^2 \\ \frac{\partial f}{\partial y} = 2ty \end{cases} \Rightarrow$ Cont's everywhere around $(0, A)$

$\overset{\text{Picard}}{\Longrightarrow}$ A unique solution exists only on some interval around $t_0 = 0$ and not on $(-\infty, \infty)$

* To find the largest interval of existence we need to solve the equation and find y:

$$\frac{dy}{y^2} = t\,dt \Rightarrow y(t) = \frac{-1}{\frac{1}{2}t^2 + C}$$

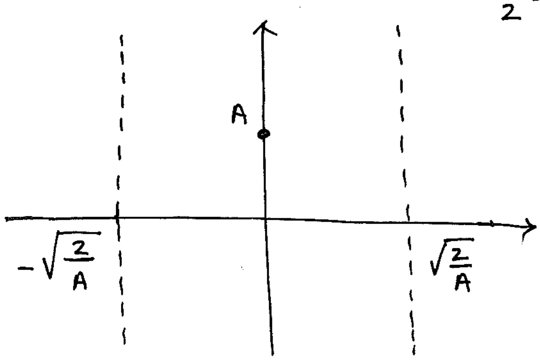
$$\xrightarrow{y(0) = A} A = \frac{-1}{C} \Rightarrow C = -\frac{1}{A}$$

Different situations : if $A<0$ then $C>0$ and $y(t) = \dfrac{-1}{\frac{1}{2}t^2+C}$ is everywhere defined

so $y(t) = \dfrac{-1}{\frac{1}{2}t^2 - \frac{1}{A}}$ is a solution on $(-\infty, \infty)$.

if $A>0$ then $C<0$ and $y(t) = \dfrac{-1}{\frac{1}{2}t^2+C}$ becomes undefined at $t = \pm\sqrt{-2C}$

$= \pm\sqrt{\frac{2}{A}}$



with this initial condition, when A>0, the largest interval of existence for the solution is (-sqrt(2/A), sqrt(2/A))

if $A=0 \Rightarrow$ NO solution of the form $y(t) = \dfrac{-1}{\frac{1}{2}t^2+C}$, only solution is $y=0$.

- Statements such as "Show/prove there exists a unique solution" suggest Picard's Thm.

- If Picard's Thm fails (f or $\frac{\partial f}{\partial y}$ discont's) does NOT mean there is NO solution.

  It should be considered case by case.

Practice. Show that the IVP $\dfrac{dy}{dt} = \dfrac{t-3}{(y-1)^3}$, $y(0)=0$ has a unique solution.

What about the same equation when $y(0)=1$?

Practice. Consider $t x' = x + 3t^2$, what can be said about existence and uniqueness of a solution when $x(0) = 0$?

What about $x(1) = 0$?

The first two columns are included in the exam.

ODE systems $\vec{x}' = A\vec{x}$ (A: constant matrix with real entries) $A_{2\times 2}$

$\det(A - rI) = 0 \rightsquigarrow r_1, r_2$ eigval

$(A - rI)\vec{v} = 0 \rightsquigarrow v_1, v_2$

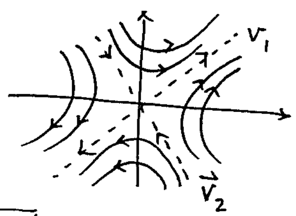
I. "A" has two real distinct eigval with two linearly indep. eigvec.

$r_1 \rightsquigarrow \vec{v}_1 \rightsquigarrow \vec{x}_1(t) = \vec{v}_1 e^{r_1 t}$

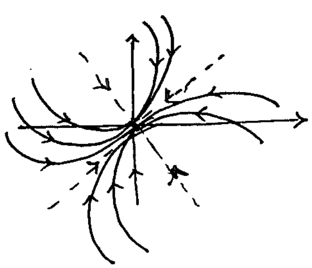
$r_2 \rightsquigarrow \vec{v}_2 \rightsquigarrow \vec{x}_2(t) = \vec{v}_2 e^{r_2 t}$

$\Rightarrow \vec{x}(t) = c_1 \vec{v}_1 e^{r_1 t} + c_2 \vec{v}_2 e^{r_2 t}$

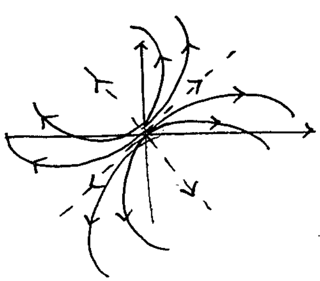
$\boxed{\text{I.1}}$ $r_1 > 0,\ r_2 < 0 \rightsquigarrow$ origin is an unstable saddle



$\boxed{\text{I.2}}$ $r_1 < 0,\ r_2 < 0 \rightsquigarrow$ Origin is an asymp. stable sink node.



$\boxed{\text{I.3}}$ $r_1 > 0,\ r_2 > 0 \rightsquigarrow$ Origin is an ustable source node.



II. "A" has two complex conjugate eigval $r_{1,2} = a \pm ib$

$\vec{v}_{1,2} = v_1 \pm i v_2$

Only one of them is enough

$\Rightarrow \vec{x}_1(t) = \vec{v}_1 \cdot e^{(a+ib)t}$

$= \vec{v}_1 e^{at}(\cos bt + i \sin bt)$

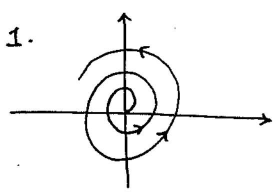
distribute $\vec{v}_1$ & simplify $= \vec{U}(t) + i\vec{V}(t)$

general soln $\Rightarrow \vec{x}(t) = c_1 \vec{U}(t) + c_2 \vec{V}(t)$

$\boxed{\text{II.1}}$ $a > 0 \rightsquigarrow$ origin is an unstable spiral point.

$\boxed{\text{II.2}}$ $a < 0 \rightsquigarrow$ origin is an asymp. stable point.

1.



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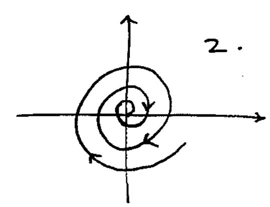
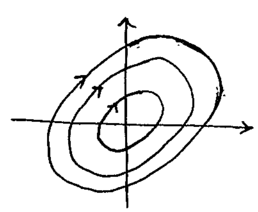


$\boxed{\text{II.3}}$ $a = 0 \rightsquigarrow$ purely imaginary

$\rightsquigarrow$ origin is a stable center. (ellipses)



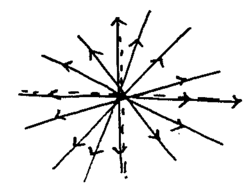
III. "A" has one real repeated eigval.

$\boxed{\text{III.1}}$ one repeated with two linearly indep eigvec:

$r \to \vec{v}_1,\ \vec{v}_2 \Rightarrow x(t) = c_1 e^{rt}(\vec{v}_1 + \vec{v}_2)$

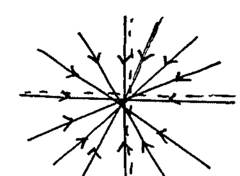
$= c_1 \vec{v}_1 e^{rt} + c_2 \vec{v}_2 e^{rt}$

$r > 0$



unstable

$r < 0$



asymp. stable

*when $A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ proper node (star poin

$\boxed{\text{III.2}}$ Defective case

one r and one v only

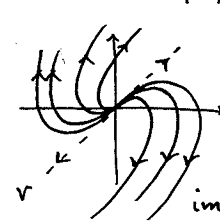
$\Rightarrow \vec{x}_1(t) = \vec{v} e^{rt} \rightsquigarrow (A - rI)v = 0$

Another vector is found by:

$(A - rI)\vec{w} = \vec{v}$

$\vec{w}$ found $\Rightarrow \vec{x}_2(t) = (t\vec{v} + \vec{w}) e^{rt}$

$\Rightarrow \vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$



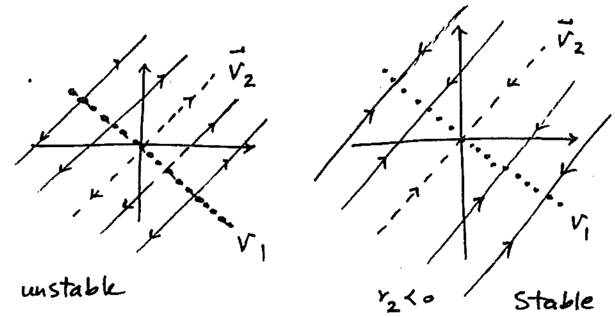
$r > 0 \rightsquigarrow$ (similar when $r < 0$)

improper node, unstable

Special case of I : when one of the eigval is $r_1 = 0$

$$\Rightarrow \quad X(t) = c_1\vec{v}_1 + c_2\,\vec{v}_2\, e^{r_2 t}$$

* Infinitely many critical points $A\vec{x}=0$, all lie on the line through $\vec{v}_1$ corresponding to $r_1 = 0$
* All other solutions are lines parallel to $\vec{v}_2$.



- You should know the shape of the trajectories and also you should be able to classify the system with vector fields when trajectories are not given.

<u>Euler Method</u>:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases} \xrightarrow{\text{Approximate}} y(T) = ?$$

$$y_{n+1} = y_n + f(t_n, y_n)\,h$$

where h = step size $= \dfrac{T - t_0}{N} = t_{n+1} - t_n$

$t_0 \quad t_1 \quad t_2 \quad \cdots \quad t_N = T$

<u>Example</u>:

$$\begin{cases} \dfrac{dy}{dt} = 0.5(1-y)\,t \\ y(0) = 2 \end{cases}$$

, approximate $y(1)$ using step size of 0.5.

$t_0 = 0$, $y_0 = 2$ $\qquad t_1 = 0.5$, $y_1 = y(0.5)$ $\qquad t_2 = 1$, $y_2 = y(1)$

$$y_1 = 2 + 0.5\,\underbrace{\big((1-2)\times 0\big)}_{f(t_0,y_0) = f(0,2)} \times 0.5 = 2 \quad \rightsquigarrow (0.5, 2)$$

$$\begin{aligned} y_2 = y(1) &= y_1 + f(t_1, y_1)\,h \\ &= 2 + 0.5\big((1-2)\,0.5\big) \times 0.5 \\ &= 2 - 0.125 = \underline{1.875} \end{aligned}$$

* Error :
  - local error in one step: $e_n \propto h^2$
  - global error $E_n \propto h$

$\Rightarrow$ Euler's method is a 1^{st} order method

<u>Improved Euler</u> :

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(t_n, y_n) + f\big(t_n, y_n + h\cdot f(t_n, y_n)\big)\Big)$$

* Error :
  - local : $e_n \propto h^3$
  - global : $E_n \propto h^2$

$\Rightarrow$ Improved Euler is a 2^{nd} order method.

Chapter Review Problems

Miscellaneous Problems. One of the difficulties in solving first-order differential equations is that there are several methods of solution, each of which can be used on a certain type of equation. It may take some time to become proficient in matching solution methods with equations. The first 24 of the following problems are presented to give you some practice in identifying the method or methods applicable to a given equation. The remaining problems involve certain types of equations that can be solved by specialized methods.

In each of Problems 1 through 24, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

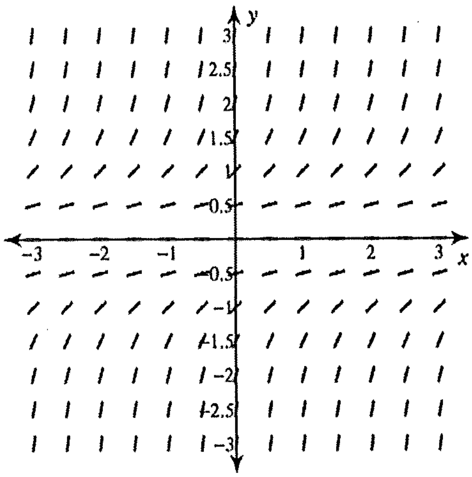
1. $\dfrac{dy}{dx} = \dfrac{x^3 - 2y}{x}$
2. $\dfrac{dy}{dx} = \dfrac{1 + \cos x}{2 - \sin y}$
3. $\dfrac{dy}{dx} = \dfrac{2x + y}{3 + 3y^2 - x}, \quad y(0) = 0$
4. $\dfrac{dy}{dx} = 3 - 6x + y - 2xy$
5. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$
6. $x\dfrac{dy}{dx} + xy = 1 - y, \quad y(1) = 0$
7. $x\dfrac{dy}{dx} + 2y = \dfrac{\sin x}{x}, \quad y(2) = 1$
8. $\dfrac{dy}{dx} = -\dfrac{2xy + 1}{x^2 + 2y}$
9. $(x^2y + xy - y) + (x^2y - 2x^2)\dfrac{dy}{dx} = 0$
10. $(x^2 + y) + (x + e^y)\dfrac{dy}{dx} = 0$
11. $(x + y) + (x + 2y)\dfrac{dy}{dx} = 0, \quad y(2) = 3$
12. $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$
13. $\dfrac{dy}{dx} = \dfrac{e^{-x}\cos y - e^{2y}\cos x}{-e^{-x}\sin y + 2e^{2y}\sin x}$
14. $\dfrac{dy}{dx} = e^{2x} + 3y$
15. $\dfrac{dy}{dx} + 2y = e^{-x^2-2x}, \quad y(0) = 3$
16. $\dfrac{dy}{dx} = \dfrac{3x^2 - 2y - y^3}{2x + 3xy^2}$
17. $y' = e^{x+y}$
18. $\dfrac{dy}{dx} + \dfrac{2y^2 + 6xy - 4}{3x^2 + 4xy + 3y^2} = 0$
19. $t\dfrac{dy}{dt} + (t + 1)y = e^{2t}$
20. $xy' = y + xe^{y/x}$
21. $\dfrac{dy}{dx} = \dfrac{x}{x^2y + y^3}$ *Hint:* Let $u = x^2$.
22. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$
23. $(3y^2 + 2xy) - (2xy + x^2)\dfrac{dy}{dx} = 0$
24. $xy' + y - y^2e^{2x} = 0, \quad y(1) = 2$

1 – 2 Write an exact equation that has the given solution. Then verify that the equation you have found is exact.

1. It has the general solution $x^2\tan y + x^3 - y^2 - 3x^4y^2 = C$.
2. It has a particular solution $2xy - \ln xy + 5y = 9$.

For each problem, find a differential equation that could be represented with the given slope field.

3)

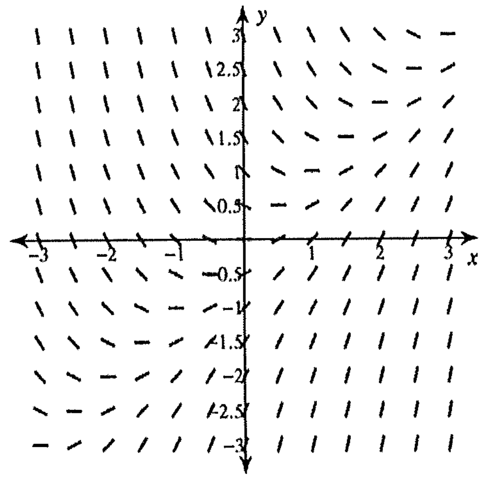


A) $\frac{dy}{dx} = -\frac{1}{x}$ B) $\frac{dy}{dx} = -\frac{1}{y}$

C) $\frac{dy}{dx} = 1$ D) $\frac{dy}{dx} = y^2$

The slope field corresponds to an autonomous equation. Test the slope for (1,0) for example to decide between B and D

4)



A) $\frac{dy}{dx} = x + y$ B) $\frac{dy}{dx} = x - y$

C) $\frac{dy}{dx} = xy$ D) $\frac{dy}{dx} = -xy$

Test for example (1,1) at which the slope is zero. Only B works.

1 – 8 Find and classify all equilibrium solutions of each equation below.

1. $y' = 100y - y^3$

2. $y' = y^3 - 4y$

3. $y' = y(y-1)(y-2)(y-3)$

4. $y' = \sin y$

5. $y' = \cos^2(\pi y/2)$

6. $y' = 1 - e^y$

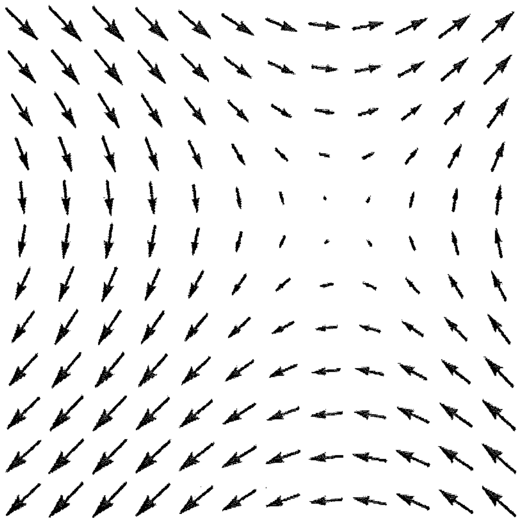
7. $y' = (3y^2 - 2y - 1)e^{-2y}$

8. $y' = y(y-1)^2(3-y)(y-5)^2$

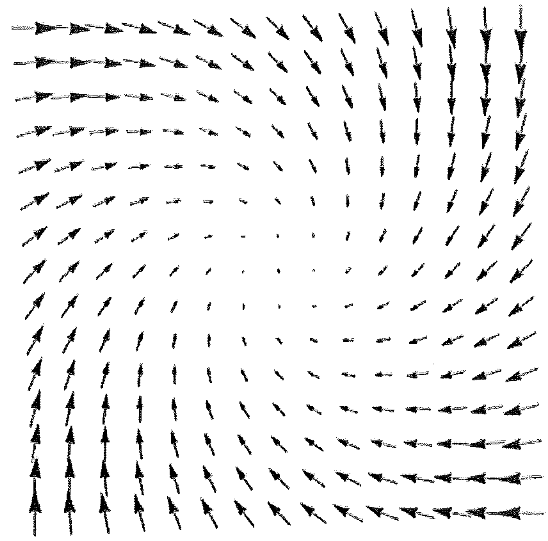
Answers A-2.1:

1. $y = 0$ (unstable), $y = \pm 10$ (asymptotically stable)
2. $y = 0$ (asymptotically stable), $y = \pm 2$ (unstable)
3. $y = 0$ and $y = 2$ (asymptotically stable), $y = 1$ and $y = 3$ (unstable)
4. $y = 0, \pm 2\pi, \pm 4\pi, \ldots$ (unstable), $y = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ (asymptotically stable)
5. $y = \pm 1, \pm 3, \pm 5, \ldots$ (all are semistable)
6. $y = 0$ (asymptotically stable)
7. $y = -1/3$ (asymptotically stable), $y = 1$ (unstable)
8. $y = 0$ (unstable), $y = 1$ and $y = 5$ (semistable), $y = 3$ (asymptotically stable)

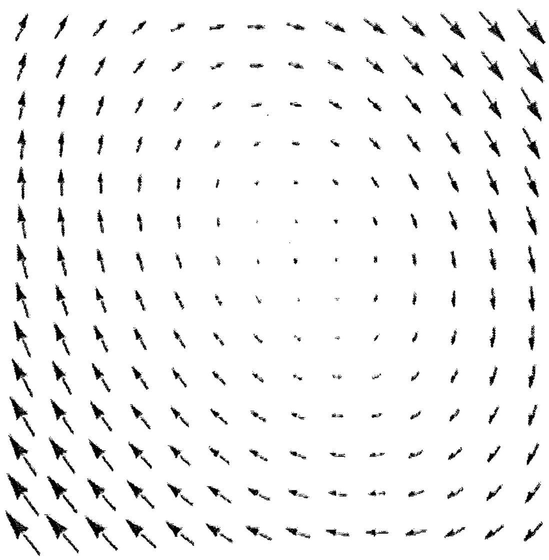
saddle, unstable, two real distinct
eigenvalues with opposite signs



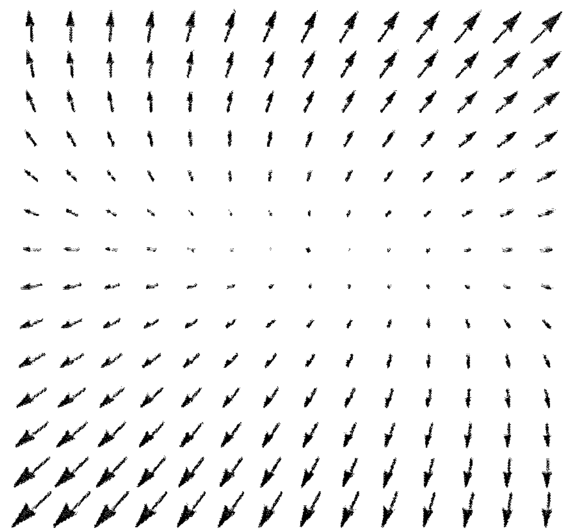
spiral sink, asymp. stable, complex
eigvalues with negative real parts

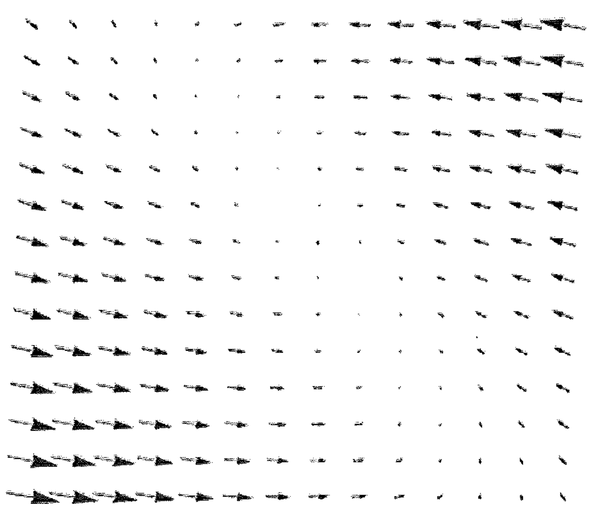


center, stable, complex purely
imaginary eigvalues



source, unstable, two real distinct
eigenvalues both positive





sink, asymp. stable, two real
distinct eigenvalues both negative

Problems

In each of Problems 1 through 4:

G a. Draw a direction field and sketch a few trajectories.

b. Express the general solution of the given system of equations in terms of real-valued functions.

c. Describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

13. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\mathbf{x}$

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

11. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$