Sept 28
Lecture 11

Last Class :

1st-order

- System of ODEs : $\vec{x}'(t) = F(t, \vec{x}(t))$ where

$$X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$ is a vector solution of the system.

- A 2x2 linear system $\vec{x}'(t) = A(t)\vec{X}(t) + G(t)$

or $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

- In particular; autonomous 1st order linear homogeneous system : $\vec{x}'(t) = A\,\vec{X}(t)$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Solution should be of the form:

$\vec{x}(t) = \vec{v}\, e^{rt}$ $\rightsquigarrow$ solve this for $\vec{v}$ and r ?

$A\vec{v} = r\vec{v}$ Find eigval of A : r

and eigvec of A : $\vec{v}$

Example 1: $\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}(t)$

We computed two solutions :

$r_1 = 2$

$\vec{v}^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x}^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

$r_2 = -1$

$\vec{v}^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

$$\xrightarrow[\text{linearly indep (3 Facts)}]{\vec{x}^{(1)} \text{ and } \vec{x}^{(2)} \text{ are}}$$ linear combination of $\vec{x}^1$ and $\vec{x}^2$ is the general sol.

General Solution : $\vec{X}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

* c_1 and c_2 are determined by applying the initial Condition .

Fundamental set of solutions : For an nxn ODE system, if $\vec{x}^{\,1}(t), \ldots, \vec{x}^{\,n}(t)$ are n linearly independent solutions then the set $\{ \vec{x}^{\,1}(t), \ldots, \vec{x}_n(t) \}$ is called the fundamental set of sol's .

In Example 1 : $\left\{ \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}, \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix} \right\}$ is the fundamental set of sol's .

In general: $\vec{X}' = AX$ $\circledast$ $\qquad \dfrac{dX}{dt} = AX$

Case 1 : "A" has two distinct real eigval r_1 and r_2 with corresponding eigvec $\vec{V}_1$ and $\vec{V}_2$ then a general sol'n to $\circledast$ has the following form:

$$\vec{X}(t) = c_1 \; \vec{X}^{(1)}(t) + c_2 \; \vec{X}^{(2)}(t) \qquad \text{where}$$

$$\vec{x}^{\,1}(t) = \vec{v}_1 \, e^{r_1 t} \qquad \text{and} \qquad \vec{x}^{\,2}(t) = \vec{v}_2 \, e^{r_2 t}$$

With initial Condition given as

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

c_1 and c_2 will be determined .

* Please Forgive my mess-up with indices

* I try to use superscript for vectors , but please pay attention to context more than my notation .

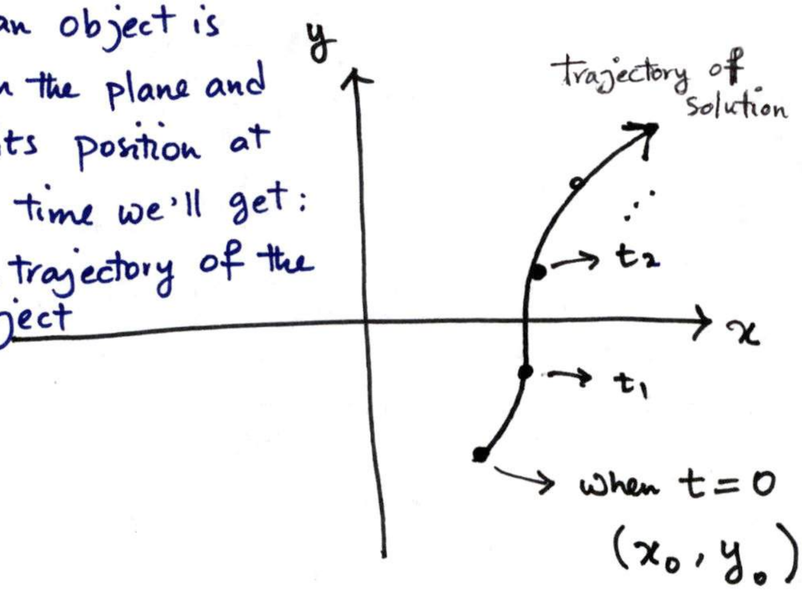
Today : Visualizing the solutions and their behaviour.

$$\vec{x}' = A\vec{x}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Solution: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a vector with two components.

Suppose an object is moving on the plane and we track its position at different time we'll get:
⇝ the trajectory of the object



2D plane of one component of the sol'n vs. the other component.

↳ 2D plane is called a phase plane

track the solution at different times, the curve obtained is the trajectory of the solution.

- Direction field for $\vec{x}' = A\vec{x}$.

Evaluate RHS at different x and y : $\begin{pmatrix} x \\ y \end{pmatrix}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} x' \\ y' \end{pmatrix} \to \text{tangent vector} \begin{array}{l} \nearrow \text{magnitude} \\ \searrow \text{direction (arrows show up)} \end{array}$$

$$\vec{x}' = AX = \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

- $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}3\\ 2\end{pmatrix}$

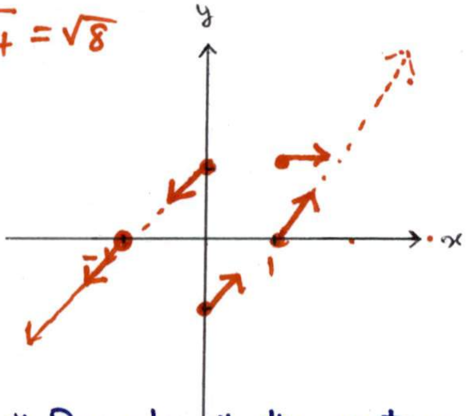
magnitute : $\sqrt{9+4} = \sqrt{13}$

- $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} \Rightarrow A\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-2\\ -2\end{pmatrix}$

$\sqrt{4+4} = \sqrt{8}$

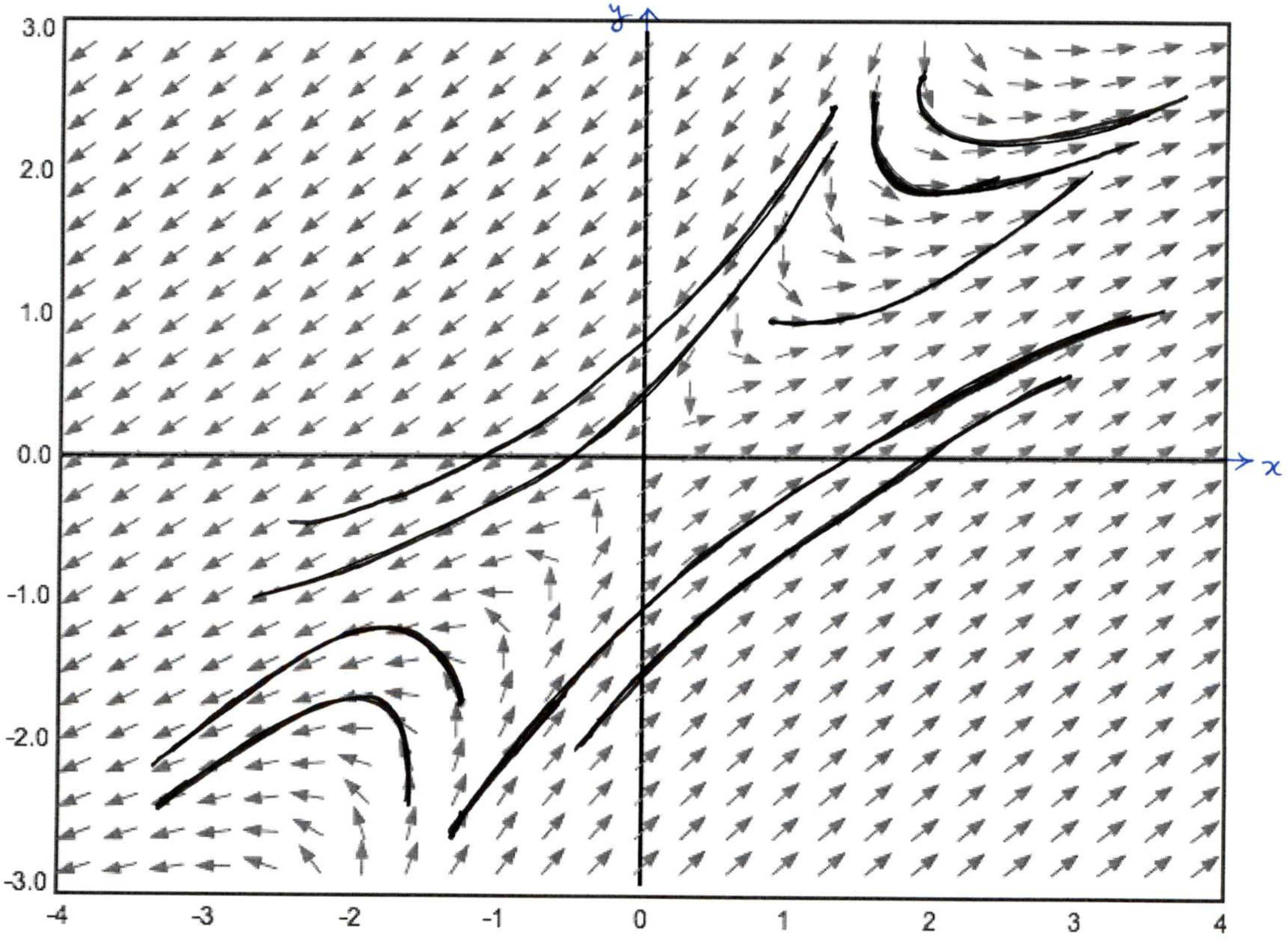
- $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-1\\ 0\end{pmatrix} \Rightarrow AX = \begin{pmatrix}-3\\ -2\end{pmatrix}$

- $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ -1\end{pmatrix} \Rightarrow AX = \begin{pmatrix}2\\ 2\end{pmatrix}$
- $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} \Rightarrow \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$



* Rescale all the vectors and just Keep their direction to avoid a bunch of overlapping vectors.

Pick some initial points and follow the tangent vector to get some trajectories. You'll get a phase portrait then.



Stability of the Solutions for $X' = AX$

$A_{2\times 2}$: → critical points are the point $\begin{pmatrix} x \\ y \end{pmatrix}$ at which

$$AX = 0 \quad (**)$$

Assume A is nonsingular $(\det A \neq 0)$

So the only solution of $**$ is $X = 0$

So $\boxed{\text{Origin is the only critical point}}$:

Case 1 . "A" has two distinct real eigval with different signs .

Example 1 : $X' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} X$

$r_1 = 2 \rightsquigarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$r_2 = -1 \rightsquigarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\rightsquigarrow \vec{X}(t) = c_1 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}}_{2y = x} + c_2 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}}$$

$$\begin{pmatrix} \overbrace{c_1 2 e^{2t}}^{x} \\ \underbrace{c_1 e^{2t}}_{y} \end{pmatrix} \qquad \begin{pmatrix} c_2 e^{-t} \\ 2 c_2 e^{-t} \end{pmatrix}$$

$\vec{v}_1$ lies on the line $x = 2y$

$\vec{v}_2$ lies on the line $y = 2x$

Observations :

(1) As $t \to \infty$, $X(t) \longrightarrow \infty$

Let's study the solution deeper and sketch its trajectories.

$$X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

(2) If a solution starts at an initial point on $\vec{v}_1$ then since

$$X(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{X(0) \text{ on } \vec{v}_1} c_2 = 0$$

So $X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ this means for all t solution will remain on $\vec{V}_1$ and as $t \to \infty$, $X(t) \longrightarrow \infty$ on $\vec{v}_1$ so we'll have outward arrows on $\vec{r}_1$.

(3) Similarly, if a solution starts at an initial point on $\vec{V}_2$ then $c_1 = 0$ and $X(t) = c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ always remains on $\vec{V}_2$ for all t So as $t \to \infty$; $X(t) \longrightarrow 0$ i.e. inward arrows on $\vec{V}_2$.

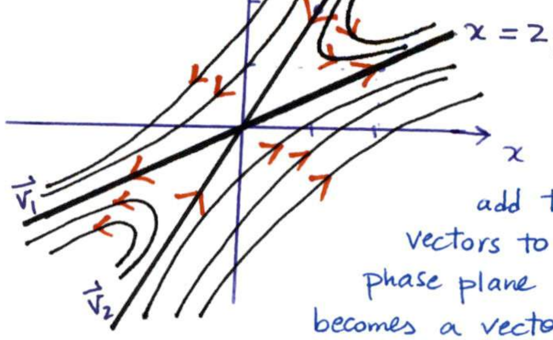
(4) For solutions starting at other initial points since e^{2t} is the dominant term as $t \to \infty$: (I) $X(t) \longrightarrow \infty$ so outward arrows on other solutions.

(II) e^{-t} term becomes negligible as $t \to \infty$

so $X(t) \approx c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$. This implies that as solutions go away from the origin when $t \to \infty$ they get close to the line through $\vec{v}_1$ i.e. $x = 2y$

* In this case: Origin is an unstable critical point since almost all solutions move away from the origin.

* This picture represents a saddle point.



add tangent vectors to this phase plane and it becomes a vector field for a saddle.

⊛ A linear system whose trajectories show the general features of those in the last diagram is said to be a saddle and the critical point is called the saddle point.

Saddle points are always unstable because almost all trajectories depart from them as t increases. It is called saddle because of its resemblance to the level curves of a saddle-shaped surface in 3D space ⇝ Hyperbola-like trajectories

Case 2 : Two eigvalues : real, distinct, both negative

Example 2 : $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$.

A has two eigval r_1, r_2 with corresponding eigvec $\vec{v}_1$, $\vec{v}_2$:

$$r_1 = -1 \;,\; \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad r_2 = -2 \;,\; \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow X(t) = \underbrace{c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}}_{\text{line : } y=0} + \underbrace{c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}}_{\text{line : } y=x}$$

* Add tangent vectors (small arrows) to this diagram and it becomes a sink vector field.

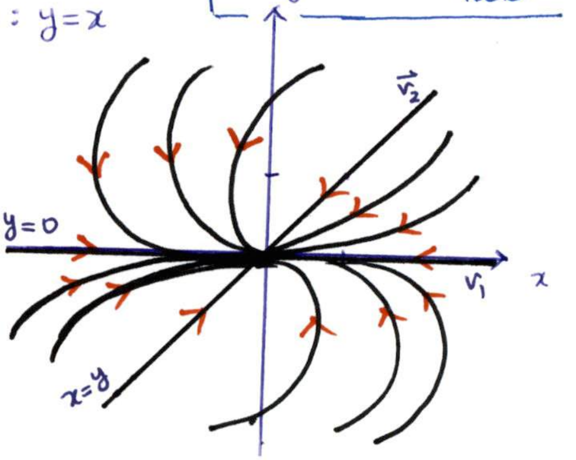
(1) $X(t) \longrightarrow 0$ as $t \rightarrow \infty$

(2) Start on $\vec{v}_1$ then $X(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$ for all t

(3) Start on $\vec{v}_2$ then $X(t) = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$ " " t.

(4) As $t \rightarrow \infty$, e^{-2t} becomes negligible compared to e^{-t} so as solutions move toward the origin they get close the line through $\vec{v}_1$ i.e. trajectories become tangent to the line $y = 0$.

* Parabola-like trajectories



* Origin is an asymptotically stable critical point.

* This case represents a sink node vector field.

↳ all trajectories approach the critical point as t increases.

Case 3 : Two eigval : real, distinct, both positive

Example 3 : $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\qquad \vec{x}' = A\vec{x}$

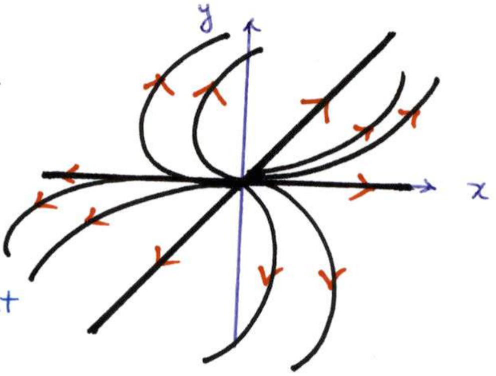
$$r_1 = 1 , \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \quad x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$r_2 = 2 , \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* Note that this system is exactly the system in Example 2 if we consider negative t ($t \to -\infty$), so as time goes forward ($t \to \infty$) we'll have exactly the same solution trajectories but they all depart from the origin as $t \to \infty$. (Outward arrows.)

* In this case, origin is an unstable critical point.

* This vector field represents a Source node. (tangent vectors must be added.)



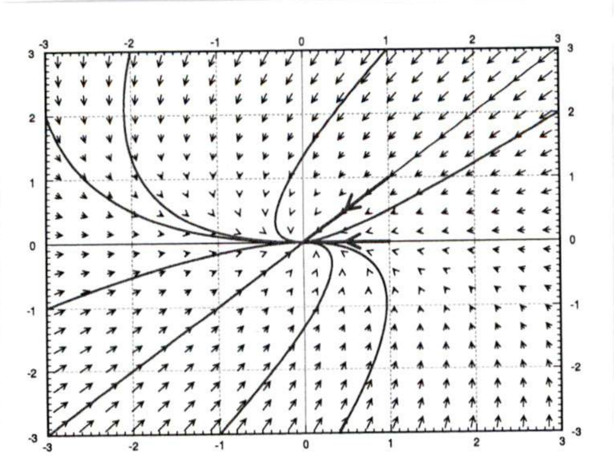
Review of Terminology

$\vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ a solution to $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

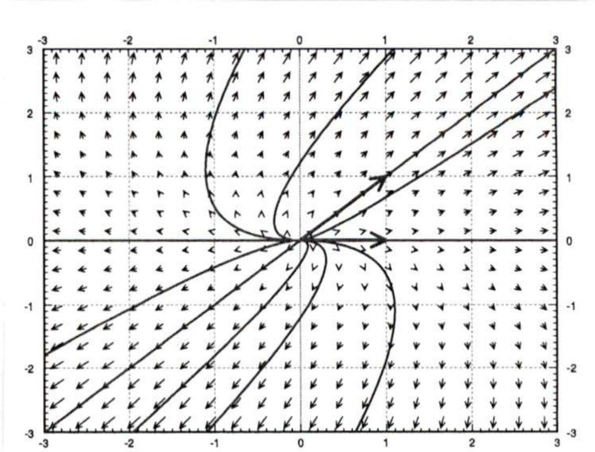
- Phase plane : x − y plane (one component of $\vec{x}$ vs. the other.)
- Trajectory : The curve that traces the position $(x(t), y(t))$ as time t varies gives a trajectory of the solution.
- Direction field : Collection of vectors (tangent vectors) obtained from evaluating RHS of equation. Vectors are rescaled in terms of their magnitude.

Phase portrait : A collection of trajectories of the solutions of the system on the phase plane as representatives of the behaviour of the system.

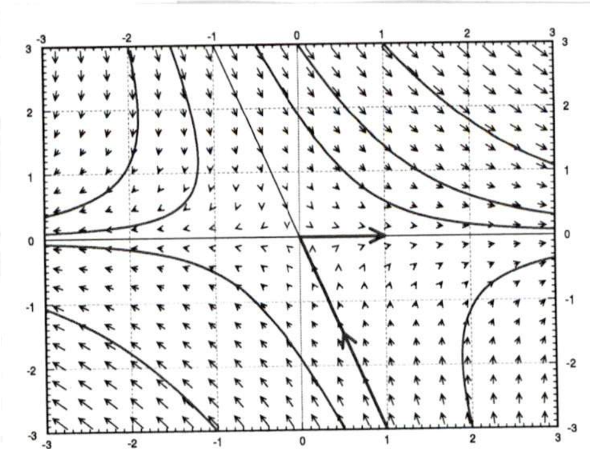
Vector fields : Vector fields and direction fields are sometimes used interchangably in some textbooks. But vector fields contain more information than dir fields. The eigenvector are specially sketched in a vector field plus the tangent vectors (magnitude and direction) at different points and usually the trajectories are sketched with arrows determining the stability of the origin and behaviour of trajectories with respect to eigvectors.



Sink node vector field



Source node vector field



Saddle vector field