Last Class :

$\vec{x}' = A\vec{x}$

A: constant matrix

System of ODEs: 1st order linear homogeneous with constant coefficients.

2x2 case : $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\begin{cases} x'(t) = ax + by \\ y'(t) = cx + dy \end{cases}$$

→ **General Solution** : (2x2)

$$\vec{X}(t) = c_1 \vec{v_1} e^{r_1 t} + c_2 \vec{v_2} e^{r_2 t}$$

where r_1 and r_2 are eigenvalues of A and $\vec{v_1}$ and $\vec{v_2}$ their corresponding eigenvectors. $(r_1 \neq r_2)$

- In nxn system :

$$\vec{X}(t) = c_1 \vec{v_1} e^{r_1 t} + c_2 \vec{v_2} e^{r_2 t} + \dots + c_n \vec{v_n} e^{r_n t}$$

where $r_1, r_2, \dots, r_n$ are eigvalues of A and $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ their corresponding eigvectors.

→ Critical point of the system $\vec{x}' = A\vec{x}$:

$$A\vec{x} = 0 \xrightarrow{\det(A) \neq 0} x = 0 \text{ the only critical point.}$$

→ Stability of the origin and visualizing the trajectories of the solution :

Case I : "A" has two distinct real eigenvalues : r_1 and r_2

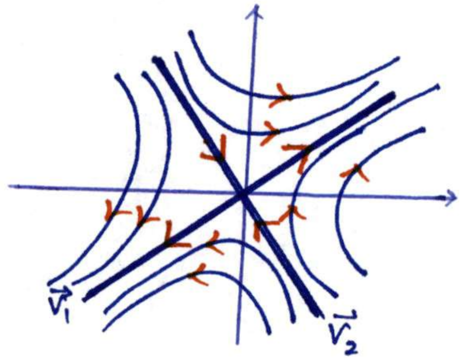
$\vec{x}_1 = \vec{v}_1 e^{r_1 t}$, $\vec{x}_2 = \vec{v}_2 e^{r_2 t}$

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$\boxed{\text{I.1}}$ r_1 and r_2 have opposite signs.

$r_1 > 0$ and $r_2 < 0$

x_1 grows $\quad$ x_2 decays

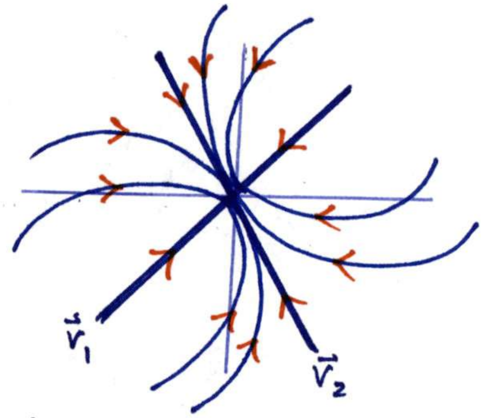


* Almost all solutions diverge from the origin ⇝ Origin is unstable

⇝ Origin is a saddle point (hyperbola-like trajectories)

$\boxed{\text{I.2}}$ r_1 and r_2 both negative

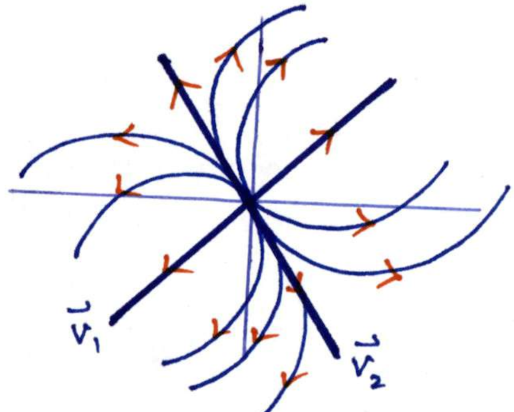
$$\underbrace{r_1 < 0 \ , \ r_2 < 0}_{\text{both decaying}}$$



* Origin is asymptotically stable
* Origin is a sink node. (parabola-like trajectories.)

$\boxed{\text{I.3}}$ r_1 and r_2 both positive

$$\underbrace{r_1 > 0 \ , \ r_2 > 0}_{\text{both growing}}$$



⇝ Exactly the same situation as I.2 just change the direction of arrows.

* Origin is unstable.
* Origin is a source node.

Note that the parabola or hyperbola shapes come from tracking the position function

$$X(t) = \begin{pmatrix} c_1 v_1^1 e^{r_1 t} + c_2 v_1^2 e^{r_2 t} \\ c_1 v_2^1 e^{r_1 t} + c_2 v_2^2 e^{r_2 t} \end{pmatrix}$$ at different times and locating x and y coordinates.

* These are NOT graph of exponential functions.

Example 1. Solve $\begin{cases} \vec{x}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \vec{x} \\ x(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \end{cases}$

Find eigval of A:

$$\det\begin{pmatrix} -2-r & 2 \\ 2 & -5-r \end{pmatrix} = (r+2)(r+5) - 4 = r^2 + 7r + 6 = 0$$

$\Rightarrow r_1 = -1, \quad r_2 = -6$

$\Rightarrow$ Origin is stable and it is a sink node.

$$r_1 = -1 \Rightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -v_1 + 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases}$$

$\Rightarrow v_1 = 2v_2$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{V_1}$

$$\boxed{\vec{X}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}} \to \text{decay}$$

$$r_2 = -6 \Rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 2v_1 + v_2 = 0 \Rightarrow v_2 = -2v_1$$

$\Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \vec{V_2}$

$$\boxed{\vec{X}_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}} \quad \text{decay}$$

General solution $\to$ $X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$

Use $X(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ to find c_1 and c_2:

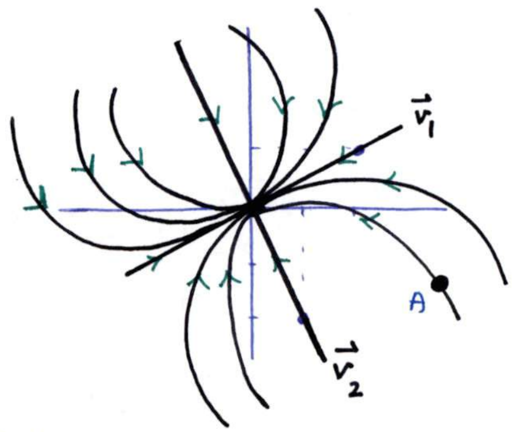
$$X(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Solve $\begin{cases} 2c_1 + c_2 = 4 \\ c_1 - 2c_2 = 0 \end{cases} \Rightarrow c_1 = \frac{8}{5}, \; c_2 = \frac{4}{5}$

$$\Rightarrow \quad X(t) = \frac{8}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \frac{4}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

$$= \begin{pmatrix} \frac{16}{5} e^{-t} + \frac{4}{5} e^{-6t} \\ \frac{8}{5} e^{-t} - \frac{8}{5} e^{-6t} \end{pmatrix}$$

The vector field for this system look like



where origin is a stable node (sink.)

The solution that we found above is one of the trajectories in the vector field.

* Also note that although these curves look like parabola, suppose starting from an initial point A, the trajectory will eventually converge to the origin, and the curve will be only one half of that parabola. The whole vector field will look like parabolas accumulating many solutions.

Example 2. Find the general solution of

$$\vec{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \vec{x}$$

$$\det\begin{pmatrix} -1-r & 2 \\ -2 & -1-r \end{pmatrix} = (1+r)^2 + 4 = r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} \quad ?!$$

No real eigenvalues

Complex eigenvalues.

Review of Complex Numbers:

$\sqrt{-1} = i$ so that $i^2 = -1$

A complex numbers looks like $z = a + ib$ where a and b are real numbers.

$a \rightsquigarrow$ real part of z : $a = \mathrm{Re}(z)$

$b \rightsquigarrow$ imaginary part of z: $b = \mathrm{Im}(z)$

Algebra:

$$(2+3i) + (-1-2i) = 1 + i$$

$$(2+3i)(-1-2i) = -2 - 4i - 3i - 6\overset{-1}{i^2} = -2 - 7i + 6$$
$$= 4 - 7i$$

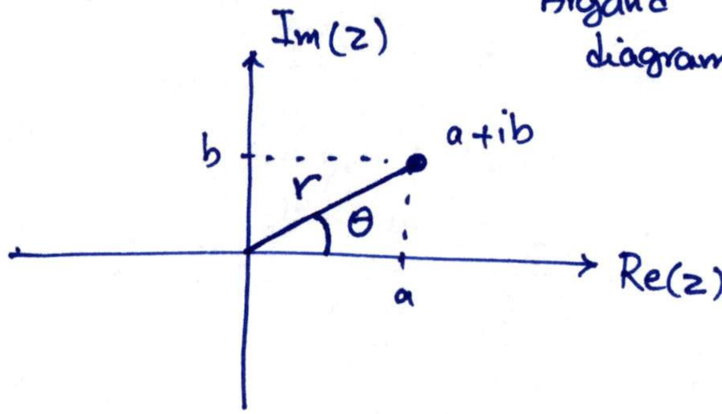
Conjugate of a complex number

$$z = a + ib \xrightarrow{\text{conjugate}} \bar{z} = a - ib$$

$$z.\bar{z} = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2$$

If $z = a + ib = (a, b)$

Argand diagram



$a = r\cos\theta$

$b = r\sin\theta$

$$\Rightarrow \quad z = a + ib = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$$

<u>Note</u> : $\boxed{r^2 = a^2 + b^2} = z\bar{z} \Rightarrow \boxed{z\bar{z} = r^2}$

$\tan\theta = \dfrac{b}{a} \Rightarrow \boxed{\theta = \arctan\left(\dfrac{b}{a}\right)}$

Polar coordinates

from Cartesian coordinates $\left\{ \begin{array}{l} (x, y) \rightsquigarrow (r, \theta) \\ (a, b) \rightsquigarrow \text{to polar.} \end{array} \right.$

$$z = r(\cos\theta + i\sin\theta)$$

r: modulus of $z = |z|$

θ: frequency of z or argument of z.

$$\boxed{\underline{\text{Euler formula}} : \quad \cos\theta + i\sin\theta = e^{i\theta}}$$

<u>Exercise</u> : Prove Euler's formula.

$$e^{2i\theta} = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$$

$$\boxed{e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned}}$$

Go back to Example 2 :

Ex 2: $\vec{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} x$

* Note: r_1 and r_2 are conjugates.

$$r = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} \begin{cases} \to r_1 = -1+2i \\ \to r_2 = -1-2i \end{cases}$$

$r_1 = -1 + 2i$ $\qquad\qquad$ $x_1(t) = \vec{v}_1 e^{r_1 t}$

$$\begin{pmatrix} -1-(-1+2i) & 2 \\ -2 & -1-(-1+2i) \end{pmatrix} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow -2iv_1 + 2v_2 = 0 \Rightarrow v_2 = +iv_1$$

$$\Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix} = \vec{v}_1$$

$$x_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+2i)t}$$

NOT a helpful form

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t} \cdot e^{2it} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t}(\cos 2t + i \sin 2t)$$

$$= e^{-t}\begin{pmatrix} \cos 2t + i \sin 2t \\ i \cos 2t - \sin 2t \end{pmatrix}$$

$$= \underbrace{e^{-t}\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}} + i \underbrace{e^{-t}\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}}$$

$$= \vec{U}(t) + i \vec{V}(t)$$

* You can verify that $\vec{U}$ and $\vec{V}$ are solutions of the equation.

$$r_2 = -1 - 2i \Rightarrow \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

*Note: v_1 and v_2 are conjugates

$$x_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-2i)t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t}\left(\cos 2t - i \sin 2t\right)$$

simplification steps :

$$= \vec{U}(t) - i\,\vec{V}(t)$$

* $\vec{x}_1$ and $\vec{x}_2$ are conjugates of each other.

$\vec{U}(t)$ and $\vec{V}(t)$ are the elements of the fundamental set of solutions : $\left\{ \vec{U}(t), \vec{V}(t) \right\}$ = fundamental set of solutions

Why are they linearly independent ?

$$\det \begin{bmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix} = e^{-2t}\cos^2 2t + e^{-2t}\sin^2 2t$$

$$= e^{-2t} \neq 0 \rightsquigarrow \vec{U}, \vec{V} \text{ linearly indep.}$$

$\vec{U}$ $\vec{V}$

Wronskian of $\vec{U}$ and $\vec{V} \neq 0$