Last Class : $\vec{x}' = A\vec{x}$ solve when A has complex eigenvalues.

Example : $\vec{x}' = \begin{pmatrix} -1 & +2 \\ -2 & -1 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

eigenvalues are :

$$\begin{cases} r_1 = -1 + 2i \\ \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{cases}$$

$$\Rightarrow \vec{x}_1(t) = \vec{v}_1 e^{r_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+2i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t} (\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} e^{-t}(\cos 2t + i \sin 2t) \\ e^{-t}(i \cos 2t - \sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos 2t \\ -e^{-t}\sin 2t \end{pmatrix} + i \begin{pmatrix} e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix}$$

$$= \vec{U}(t) + i \vec{V}(t)$$

$$\begin{cases} r_2 = -1 - 2i \\ \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases}$$

$$\Rightarrow \vec{x}_2(t) = \vec{v}_2 e^{r_2 t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-2i)t}$$

$$= \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t} (\cos 2t - i \sin 2t)$$

$$= \begin{pmatrix} e^{-t}\cos 2t - i e^{-t}\sin 2t \\ -i e^{-t}\cos 2t - e^{-t}\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}\cos 2t \\ -e^{-t}\sin 2t \end{pmatrix} - i \begin{pmatrix} e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{pmatrix}$$

$$= \vec{U}(t) - i \vec{V}(t)$$

* Note that the $\overbrace{\text{real part}}^{\vec{U}(t)}$ of X_1 (X_2) and $\overbrace{\text{imaginary part}}^{\vec{V}(t)}$ of them are themselves solutions to the system :

System : $\begin{cases} -x + 2y = x' \\ -2x - y = y' \end{cases}$ and $\vec{U}(t) = \begin{pmatrix} e^{-t}\cos 2t \\ -e^{-t}\sin 2t \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\rightsquigarrow$ Find x' and y' from $\vec{U}(t)$, plug them into the system and verify that both equations hold. Similarly for $\vec{V}(t)$.

Reminder : Lecture 11 (Friday, Sept 28) is re-uploaded with some edits. Check the new version.

General solution : $\vec{X}(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$

$$= c_1\left(\vec{U}(t) + i\vec{V}(t)\right) + c_2\left(\vec{U}(t) - i\vec{V}(t)\right)$$

$$= (c_1 + c_2)\vec{U}(t) + i(c_1 - c_2)\vec{V}(t)$$

Define: $c_1 + c_2 = k_1$ (c_1 and c_2 are some constants (complex, real))

$i(c_1 - c_2) = k_2$

This is a real solution $k_1, k_2 \in \mathbb{R}$

$$\boxed{\begin{aligned}\vec{X}(t) &= k_1\vec{U}(t) + k_2\vec{V}(t)\\ &= k_1 e^{-t}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix} + k_2 e^{-t}\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}\end{aligned}}$$

Conclusion : General sol'n involves Real part and imaginary parts of $\vec{X}_1$ (or $\vec{X}_2$) so it suffices to only calculate one eigenvector corresponding to one of the complex eigenvalues and take real and imaginary part of one solution and make a general solution with them.

$$\vec{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \rightarrow k_1 = 4, \quad k_2 = 2$$

$$\Rightarrow \quad \vec{x}(t) = 4e^{-t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + 2e^{-t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

Behaviour of the Solution

$$r_{1,2} = -1 \pm 2i = a \pm ib$$

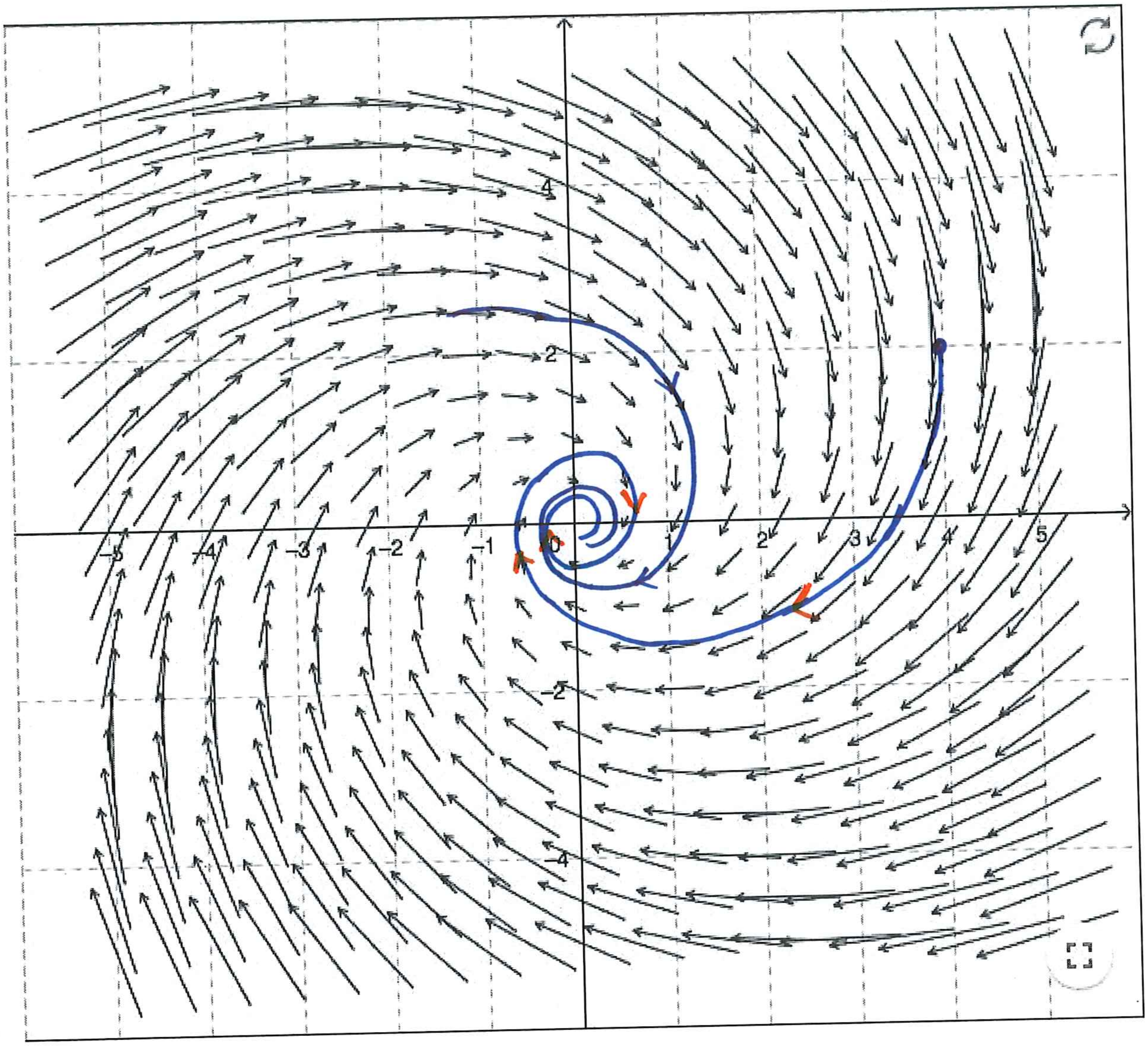
$-1 \rightarrow e^{-t}$

$2i \rightarrow \cos 2t, \ \sin 2t$

- real part of the eigenvalue controls growth / decay of the solution $\rightsquigarrow e^{at}$
  - $a > 0 \rightsquigarrow$ unstable
  - $a < 0 \rightsquigarrow$ stable (asymptotically)

- Imaginary part of r takes care of rotation and frequency of the rotation $\rightsquigarrow \sin bt$, $\cos bt$

  larger $b \rightsquigarrow$ faster rotations.

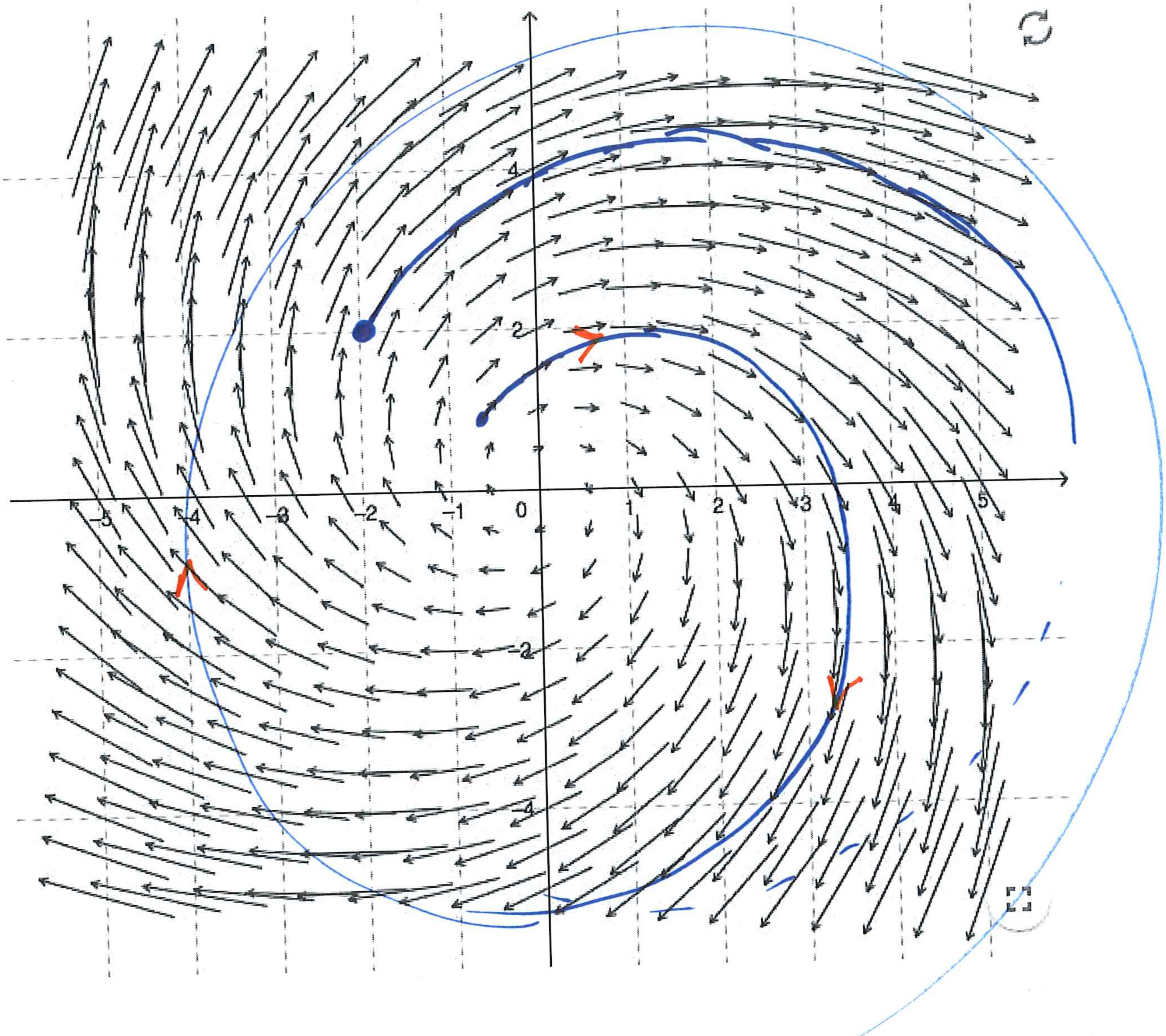


spiral trajectories, all move toward the origin

spiral sink

$\Rightarrow$ Origin is a stable (asymptotically) critical point.

$\circledast$ $r_{1,2} = a \pm ib$, $a < 0$



This is spiral trajectories moving away from the origin ⇝ spiral source

⇝ unstable

$r_{1,2} = a \pm ib$ and $a > 0$

$\underline{\underline{\text{Ex}}}$. $\vec{x}' = \begin{pmatrix} 2 & 5 \\ -1 & -2 \end{pmatrix} \vec{x}$

$\Rightarrow \quad r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad (a=0)$

purely imaginary

$r_1 = i$

$$\begin{pmatrix} 2-i & 5 \\ -1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \begin{array}{l} \to (2-i)v_1 = -5v_2 \\ \to -v_1 = (2+i)v_2 \end{array}$$

$$\begin{pmatrix} 1 \\ \frac{-2+i}{5} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 \\ -2+i \end{pmatrix} = \vec{V}$$

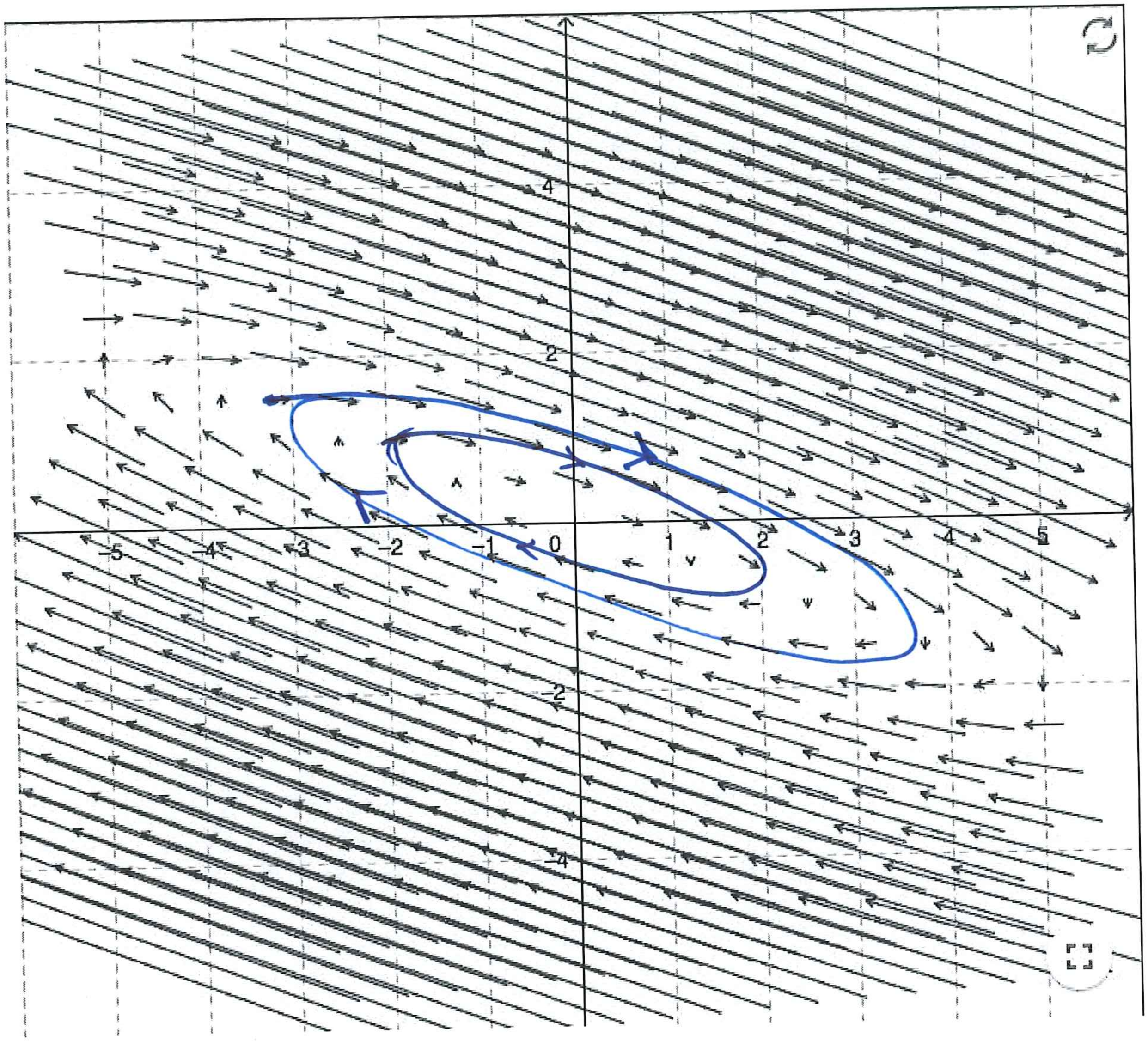
$$X_1(t) = \begin{pmatrix} 5 \\ i-2 \end{pmatrix} e^{it}$$

$$= \begin{pmatrix} 5 \\ i-2 \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

$$= \vec{U}(t) + i\vec{V}(t)$$

General Solution:

$\vec{X}(t) = k_1 \vec{U}(t) + k_2 \vec{V}(t)$ is a real solution.



This is the trajectories when $Im(r) = b$, $Re(r) = 0$ $\rightarrow r = \pm bi$

This is called a center.

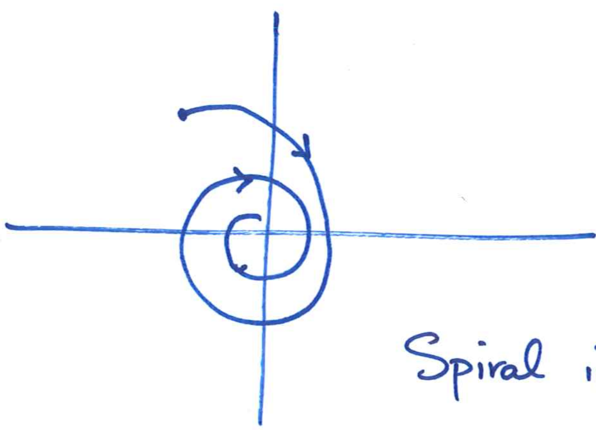
↳ This is stable but NOT asymptotically stable
(All trajectories stay within a bounded region of the phase plane as t increases but they do NOT approach the critical point (0,0).)

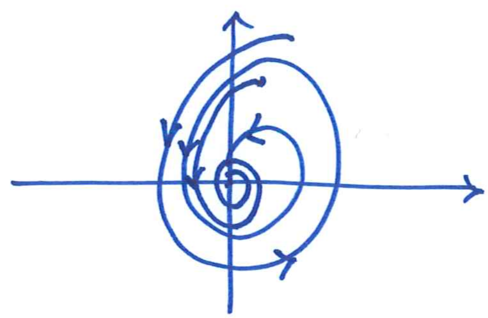
Case II : $\vec{x}' = A\vec{x}$ A: real constant matrix

and A has complex eigenvalues : $r_{1,2} = a \pm ib$

II.1 $Re(r) = a < 0 \rightsquigarrow e^{at} \rightsquigarrow$ decaying

Vector field: spiral sink $\rightsquigarrow$ stable

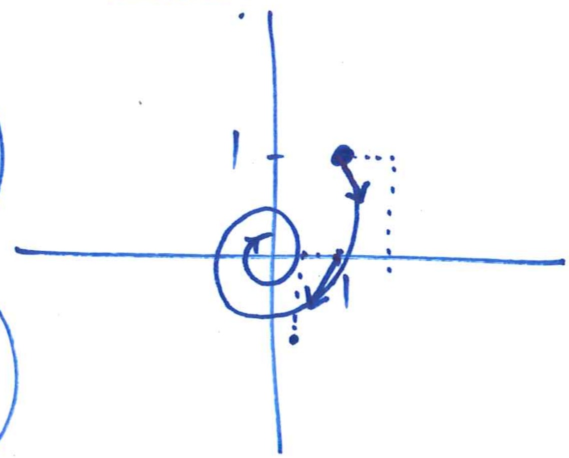




Spiral in which direction?

$$x' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

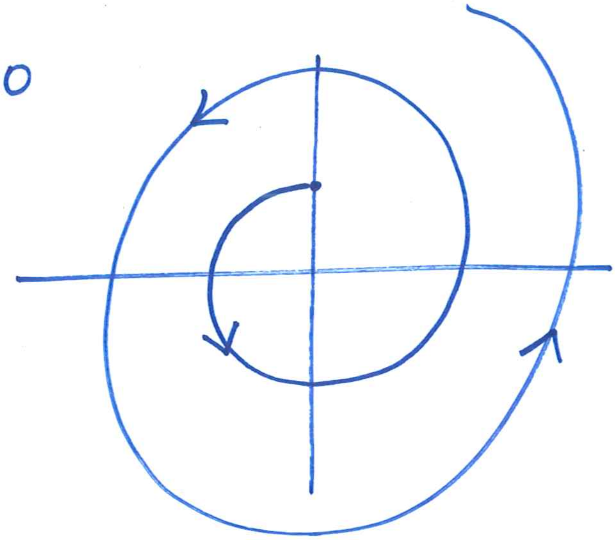


* To know the direction of the spiral, test one or two points in $X' = AX$ to find the direction of the tangent vectors.

II.2 $r = a \pm ib$ $a > 0$

spiral source

$\rightarrow$ unstable



II.3 $r = a \pm ib$ $a = 0$

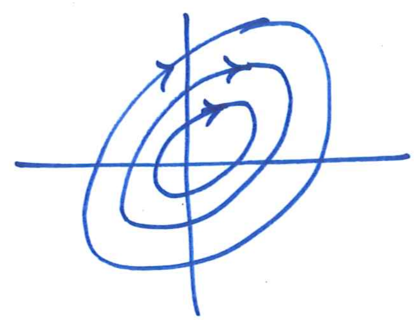
$\hookrightarrow$ purely imaginary eigval :

$\rightarrow$ ellipse : trajectories

origin is a center

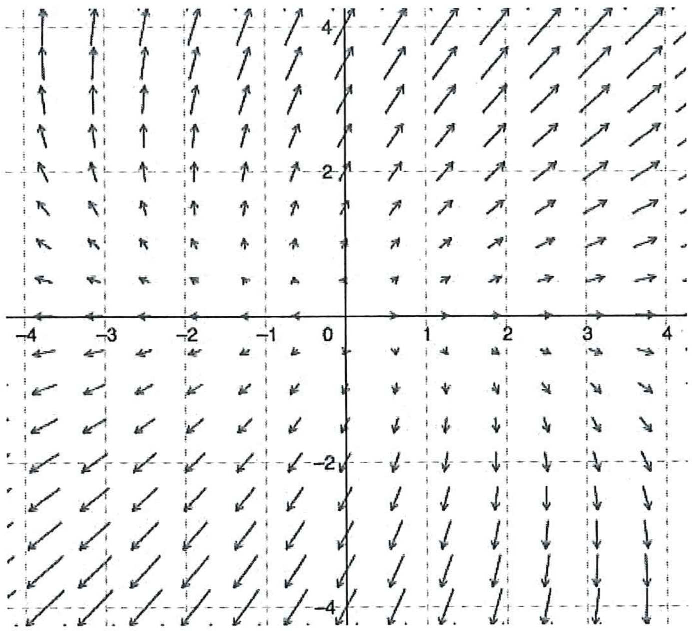
and it is stable

(NOT asymptotically)

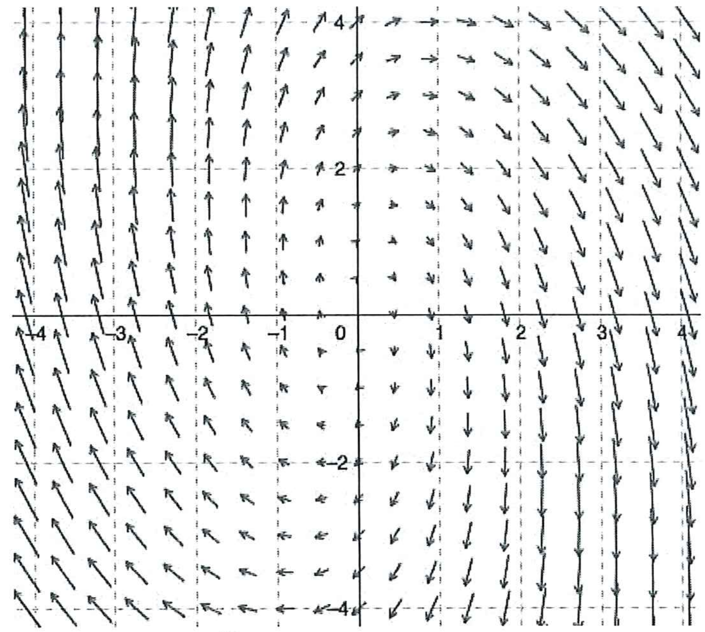


Exercise.

Match vector fields with the given systems.



A



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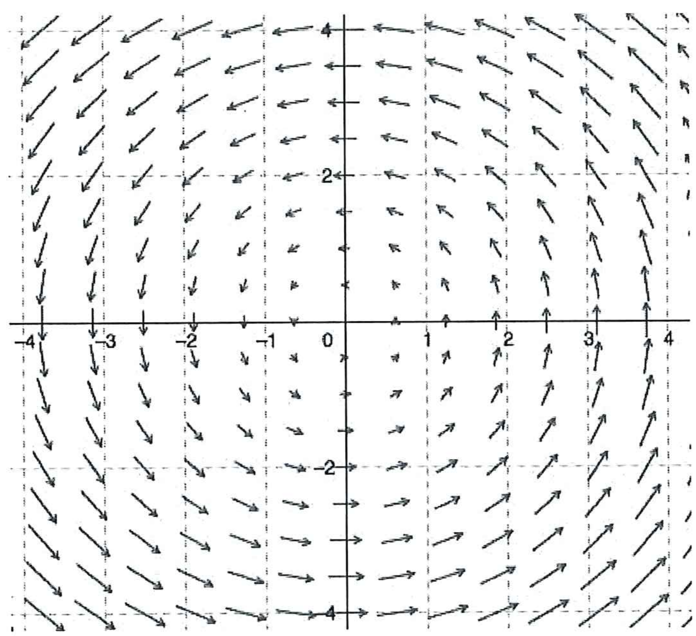
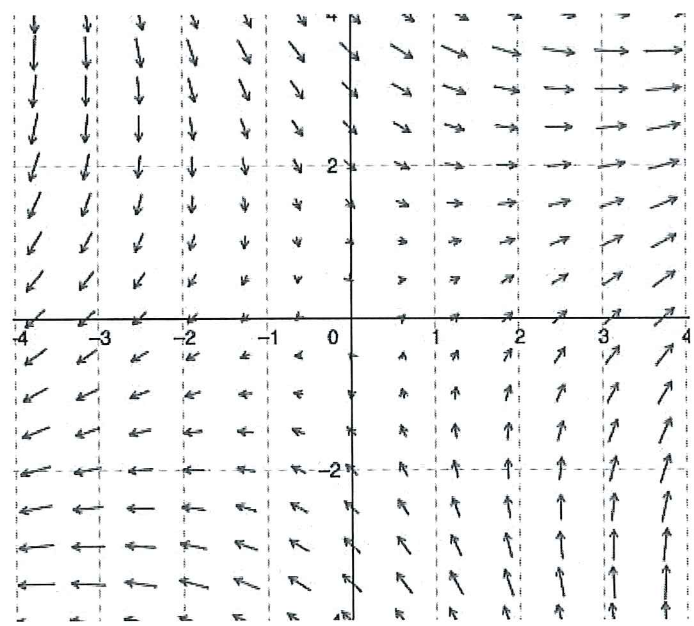


C



D

(1) $\begin{cases} x' = x+y \\ y' = x-y \end{cases}$

(3) $\begin{cases} x' = -2y \\ y' = 2x \end{cases}$

(2) $\begin{cases} x' = x+y \\ y' = 2y \end{cases}$

(4) $\begin{cases} x' = x+y \\ y' = -4x+y \end{cases}$