Oct 5
Lecture 14

Today: Repeated eigvalues

$\vec{x}' = A\vec{x}$

So far: ↳ $\det(A - rI) = 0 \xrightarrow{A_{2\times 2}}$

- two real eigval
  - → two distinct eigval
  - → $\boxed{\text{repeated eigval}}$ Today
- two complex ✓ eig $r = a \pm ib$

<u>Example 1</u>. $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

$$\det \begin{pmatrix} 2-r & 0 \\ 0 & 2-r \end{pmatrix} = 0 \Rightarrow (2-r)^2 = 0 \Rightarrow r = 2$$

repeated

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$ v_1 and v_2 are both free variables.

Just pick two <u>linearly independent</u> vectors.

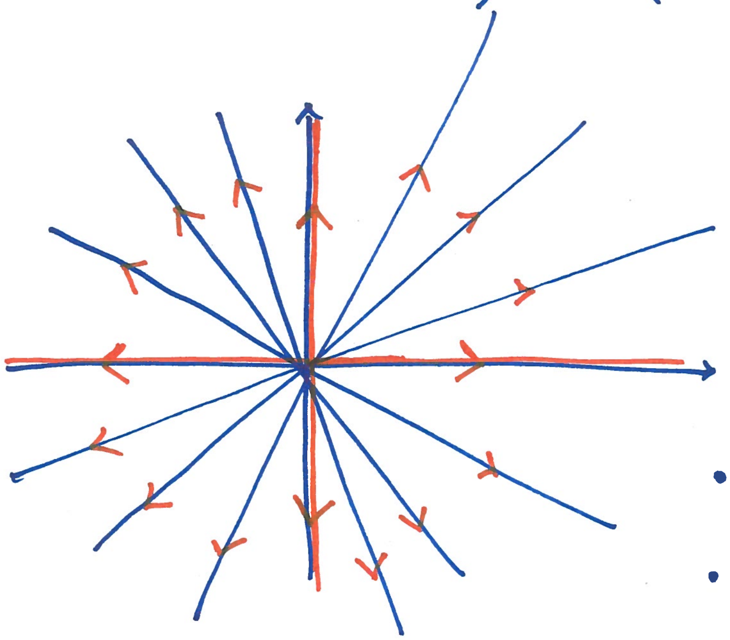
easiest case: $\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{2t}$$

$$\vec{x}(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \quad x = c_1 e^{2t} \Rightarrow e^{2t} = \frac{x}{c_1}$$

$$\Rightarrow y = c_2 e^{2t} = \frac{c_2 x}{c_1}$$

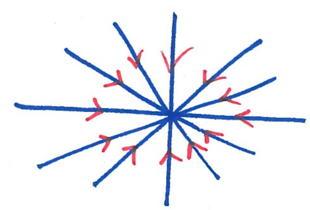
$$\Rightarrow \boxed{y = \frac{c_2}{c_1} x}$$



- Origin is unstable $(r > 0)$
- proper node

- if $r < 0 \Rightarrow \vec{x}(t) = c_1 v_1 e^{rt} + c_2 v_2 e^{rt}$

Origin is asymp. stable

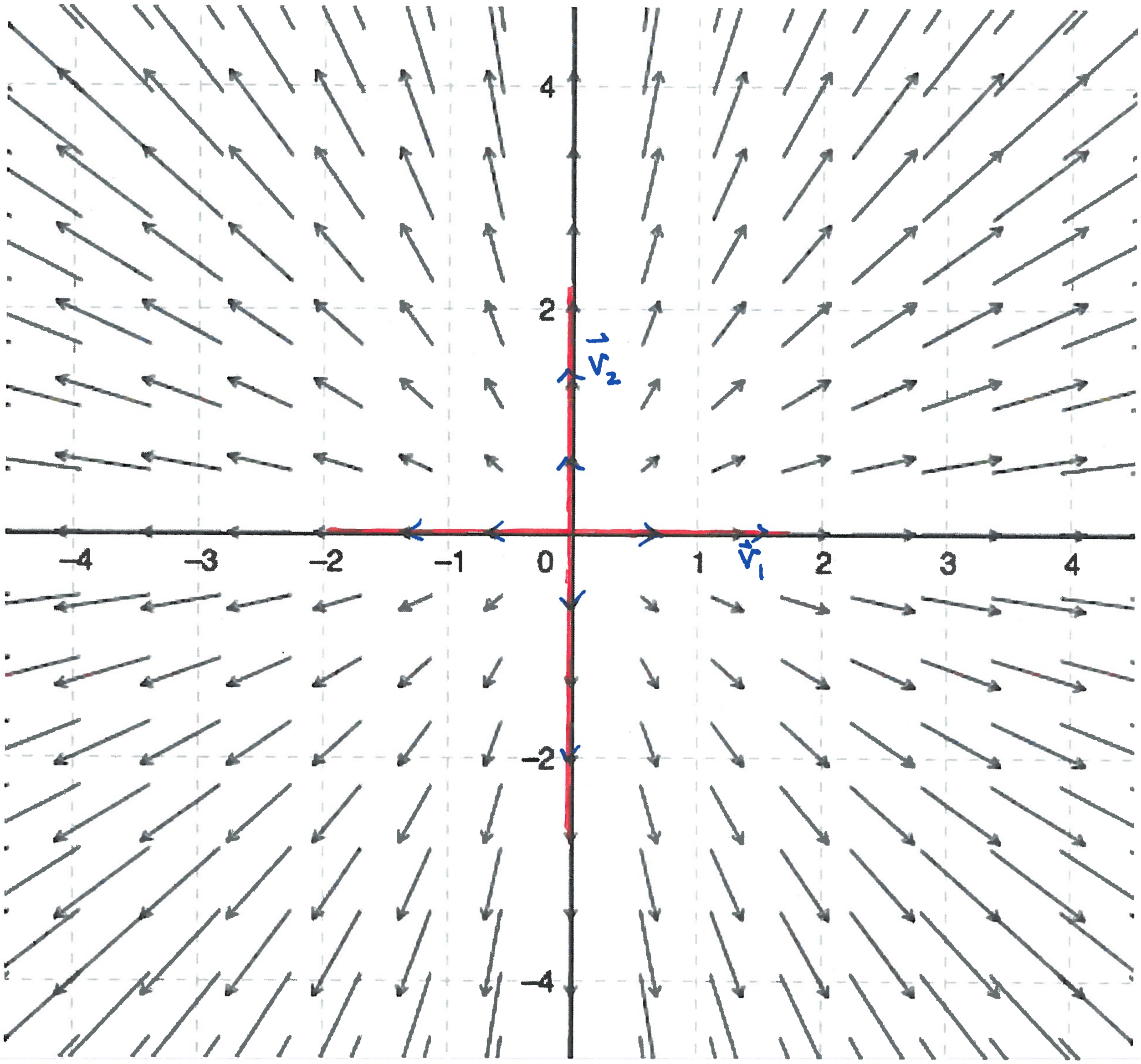


- Remark. This case happens when A is a constant multiple of I : $A = rI = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$

Ex 1 : $\begin{cases} x' = 2x \\ y' = 2y \end{cases}$ or $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x}$

One eigval $r = 2$ repeated, two linearly indep. eigvec: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General sol'n $\longrightarrow$ $x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$



* Origin is unstable
* This vector field is called a proper node or sometimes a star point.

Example 2 : $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$

$$\det \begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} = (r-2)^2 = 0 \Rightarrow r = 2 \text{ repeated}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \quad v_1 + v_2 = 0 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{V}_1 = \vec{V}$$

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

How to find a vector w which is linearly independent from $\vec{v}_1 = \vec{v}$? Generalized eigenvector

w is a vector that satisfies :

$$\boxed{(A - rI)\vec{w} = \vec{V}}$$

general eigvec

$(A - rI)\vec{v} = 0$

eigvec

Exercise

Remark : $\vec{v}$ and $\vec{w}$ are linearly independent .

Then the other solution will be :

$$\vec{X}_2(t) = (t\vec{v} + \vec{w}) e^{rt}$$

Why is $\vec{X}_2(t) = \left(t\vec{v} + \vec{w}\right)e^{rt}$ a solution?

$$\vec{X}' = AX$$

Things we know
$$\begin{cases} v: & Av = rv \\ w: & (A - rI)w = v \rightsquigarrow Aw = (rw + v) \end{cases}$$

$$\vec{X}_2(t) = (t\vec{v} + \vec{w})\,e^{rt}$$

Claim: $\vec{X}_2$ is a solution:

$$\text{LHS: } \vec{X}_2'(t) = \vec{v}e^{rt} + (t\vec{v} + \vec{w})\,r.e^{rt}$$
$$= \left(\vec{v} + tr\vec{v} + r\vec{w}\right)e^{rt}$$

$$\text{RHS: } AX_2 = A.(t\vec{v} + \vec{w})\,e^{rt}$$

$$= \left(t\,A\vec{v} + A\vec{w}\right)e^{rt}$$

$$= \left(tr\vec{v} + r\vec{w} + \vec{v}\right)e^{rt}$$

=

So $\vec{X}_2$ solves the equation.

Note that: the fact that $Av = rv$
and $Aw = (rw + v)$

helped to simplify terms and prove that X_2 is a solution.

Find w ? $\quad r = 2$

$$V = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(A - rI)\vec{w} = \vec{v}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -w_1 - w_2 = 1 \\ w_1 + w_2 = -1 \end{cases} \Rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \vec{W}$$

generalized eigvector

$$\vec{X}_2(t) = (t\vec{v} + \vec{w})e^{rt}$$

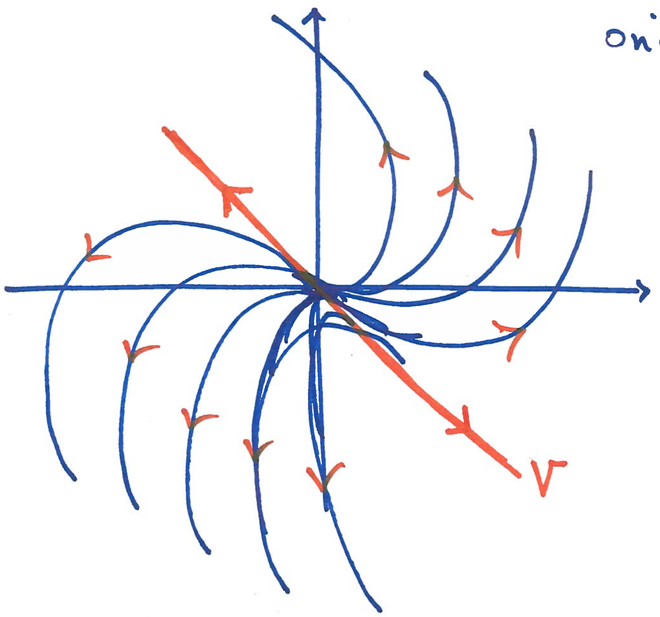
$$= \left(t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)e^{2t}$$

$$\vec{X}_2(t) = \begin{pmatrix} t \\ -t-1 \end{pmatrix} e^{2t}$$

General solution : $\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t)$

$$= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} t \\ -t-1 \end{pmatrix}e^{2t}$$

origin is unstable.



origin unstable:

improper node.

if $r<0$, then origin becomes asymptotically stable.

Case III. $\vec{x}' = A\vec{x}$

A has one repeated eigenvalue: r

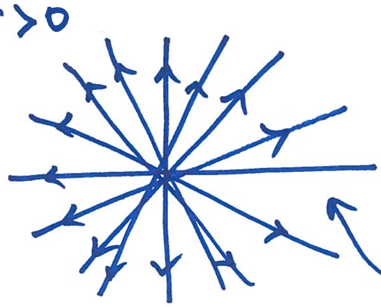
One repeated eigval but two linearly indep eigvec

$r \to \vec{v}_1$

$r \to \vec{v}_2$

$$X(t) = c_1 \vec{v}_1 e^{rt} + c_2 \vec{v}_2 e^{rt}$$

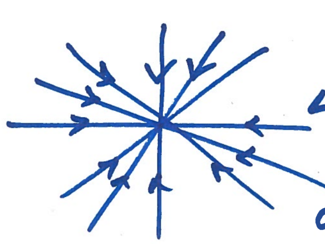
$r>0$



proper node

unstable

$r<0$



asymp. stable

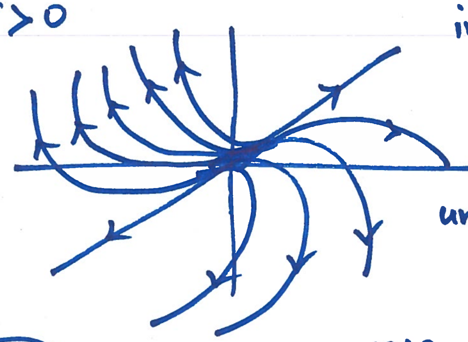
One repeated ei: r and one eigvec: v

find generalized eigvec: w

$$\vec{x}_2(t) = (t\vec{v} + \vec{w})e^{rt}$$

$$\vec{x}(t) = c_1 v e^{rt} + c_2 (t\vec{v} + \vec{w}) e^{rt}$$

$r>0$



improper node

unstable

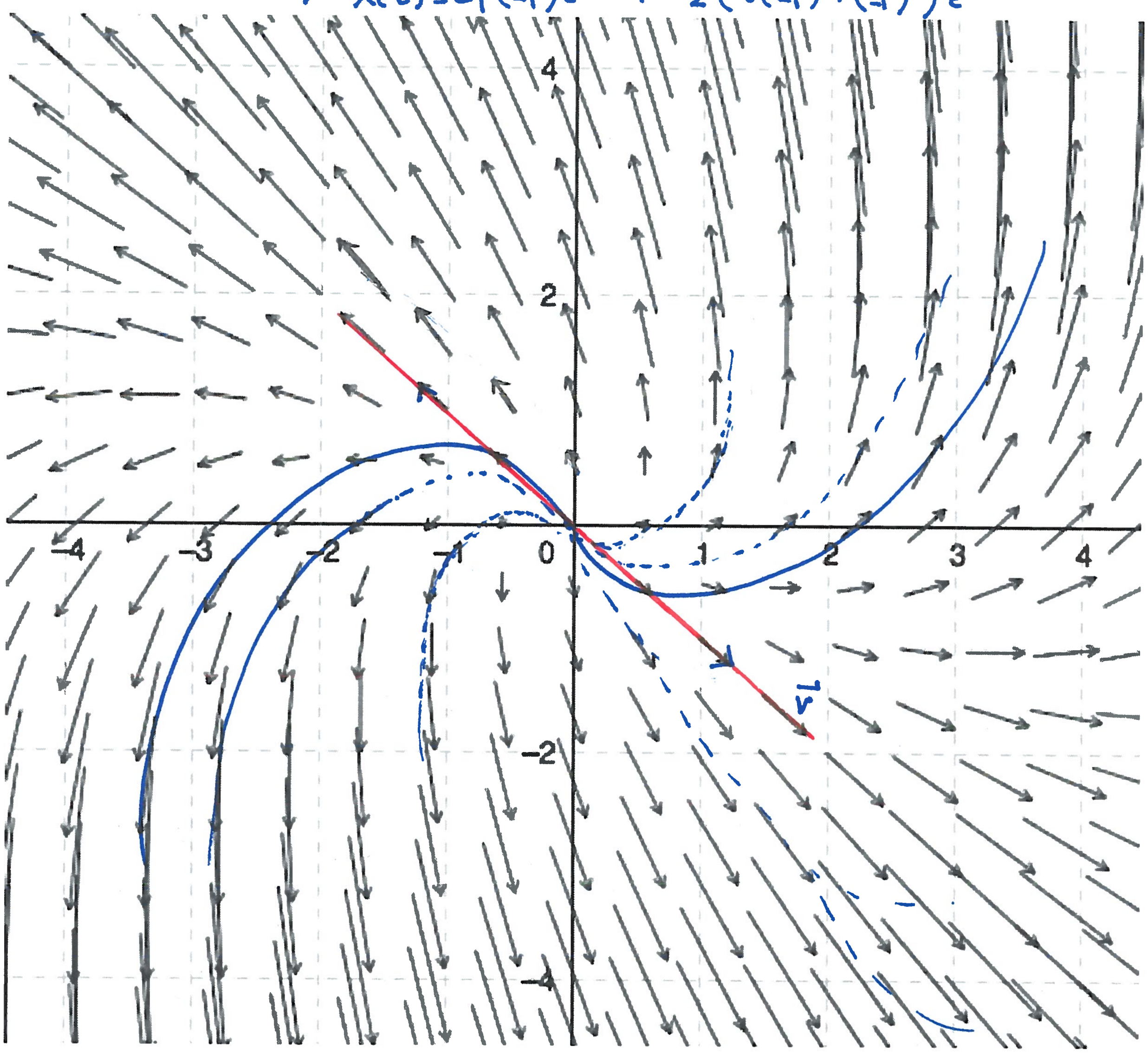
$r<0$ similar

Ex 2 : $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$ or $\begin{cases} x' = x - y \\ y' = x + 3y \end{cases}$

One eigval $r = 2$ repeated $\xrightarrow[\text{eigvec}]{\text{only 1}}$ $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\xrightarrow[\text{linear indep of } v]{\text{Need another vector}}$ $\vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ where $\vec{w}$ solves $(A - 2I)\vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \vec{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) e^{2t}$



* Origin is an unstable <u>improper node</u>.

Practice : Find the general solution to

$$\begin{cases} x' = 2x - y \\ y' = -2x + y \end{cases}$$

Find the equilibrium solution(s) (critical point) and describe the vector field and the trajectories.

eigvalues : $r_1 = 0$, $r_2 = 3$

eigvec : $\vec{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\vec{X}(t) = c_1 \vec{V}_1 e^{r_1 t} + c_2 \vec{V}_2 e^{r_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

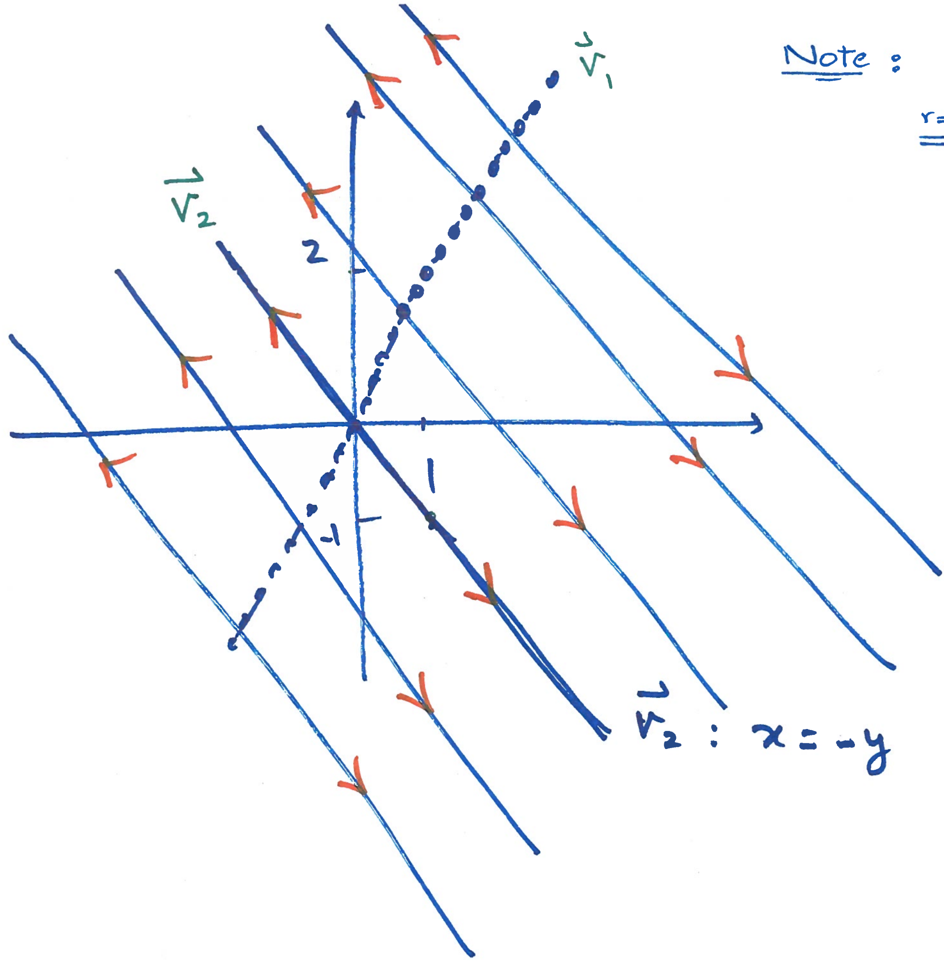
$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \boxed{\begin{array}{c} x \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \\ y \end{array}} \rightarrow x = -y$$

$$\vec{X}' = AX = \underbrace{\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}}_{A - rI} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$AX = 0$

Equilibrium solutions : all the points on the line through $\vec{V}_1$

$\rightarrow$ infinitely many solutions :



Note: $\det(A - rI) = 0 \Rightarrow r = 0,\ r = 3$

$\overset{r=0}{\Longrightarrow} (A - rI)\vec{v} = 0$

$\overset{r=0}{\Rightarrow}\ A\vec{v} = 0 \Rightarrow$ All $\vec{v}$ solving this are critical points

$\Rightarrow$ These are the dots on the line.

$\longrightarrow$ How do we get these lines all parallel to $\vec{v}_2$?

Note: $\vec{x}_2(t) = v_2 e^{r_2 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} = \begin{pmatrix} e^{3t} \\ -e^{+3t} \end{pmatrix} \rightsquigarrow$ trajectory $x = -y$

Now take the general solution:

$$\vec{x}(t) = \begin{pmatrix} c_1 + c_2 e^{3t} \\ 2c_1 - c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad e^{3t} = \frac{x - c_1}{c_2}$$

$$y = 2c_1 - \cancel{c_2} \cdot \frac{x - c_1}{\cancel{c_2}}$$

$$\Rightarrow \quad y = 2c_1 - x + c_1$$

$\longrightarrow$ line parallel to $y = -x$

So for any c_1 and c_2 trajectories are line parallel to the line that goes through $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Practice Question : $\vec{x}' = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \vec{x} = \vec{A} X$

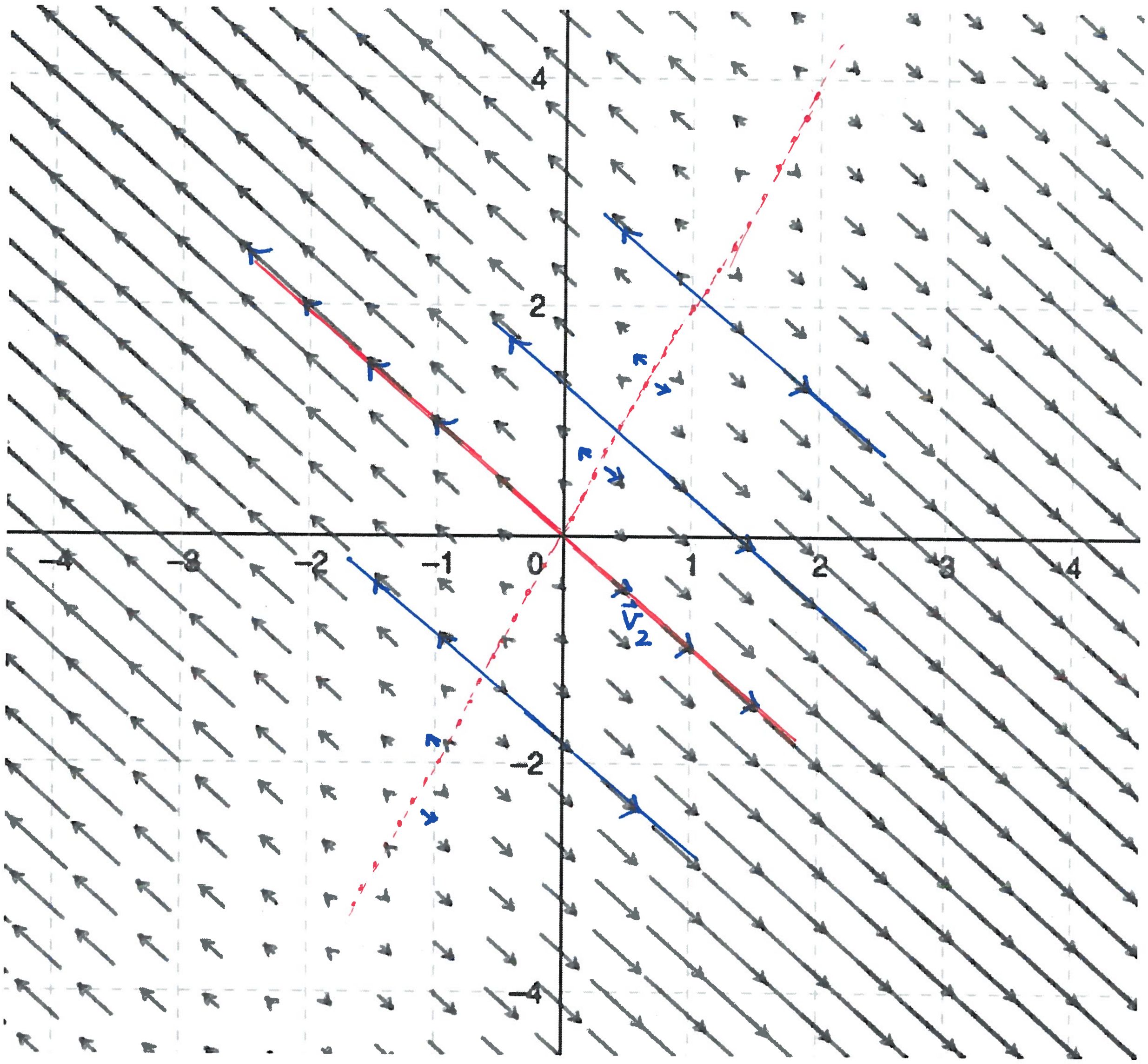
eigval : $r_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$r_2 = 3 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

* det $A = 0 \Rightarrow A$ is singular

$\Rightarrow A\vec{X} = 0$ has infinitely many solutions

$\Rightarrow$ infinitely many equilibrium poin all on the line through $\vec{v}_1$



All the critical points are unstable .