Oct 15
Lecture 17

Systems of ODE with constant coefficient and homogeneous: $\vec{X}' = A\vec{X}$

$A_{2\times 2}$: eigenvalues $r_1 \quad r_2$

$\downarrow \quad \downarrow$

eigenvectors $\vec{V}_1 \quad \vec{V}_2$

$\downarrow \quad \downarrow$

$\vec{X}_1(t) = \vec{V}_1 e^{r_1 t} \qquad \vec{X}_2 = \vec{V}_2 e^{r_2 t}$

General solution: $X(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$

if $\vec{X}(0) = \vec{X}_0 \Rightarrow c_1 \vec{X}_1(0) + c_2 \vec{X}_2(0) = \vec{X}_0$

Last day: Zero eigenvalue and Repeated eigenvalue:

Example 1. $\vec{X}' = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \vec{X} = A\vec{X}$

$$\det \begin{pmatrix} 1-r & -2 \\ 3 & -6-r \end{pmatrix} = r^2 + 5r = 0 \Rightarrow r_1 = 0, \quad r_2 = -5$$

$r_1 = 0 \xrightarrow[r=0]{(A - rI)V = 0} \underbrace{AV = 0}_{\substack{\text{has} \\ \text{infinitely many} \\ \text{solutions}}} \Rightarrow \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\det A = 0$
$\Rightarrow$ A singular

$\Rightarrow v_1 = 2v_2 \Rightarrow \vec{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

solutions $\rightsquigarrow$ all of those solutions are critical points.

$\Rightarrow \underbrace{\vec{X}_1(t) = \vec{V}_1 e^{r_1 t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

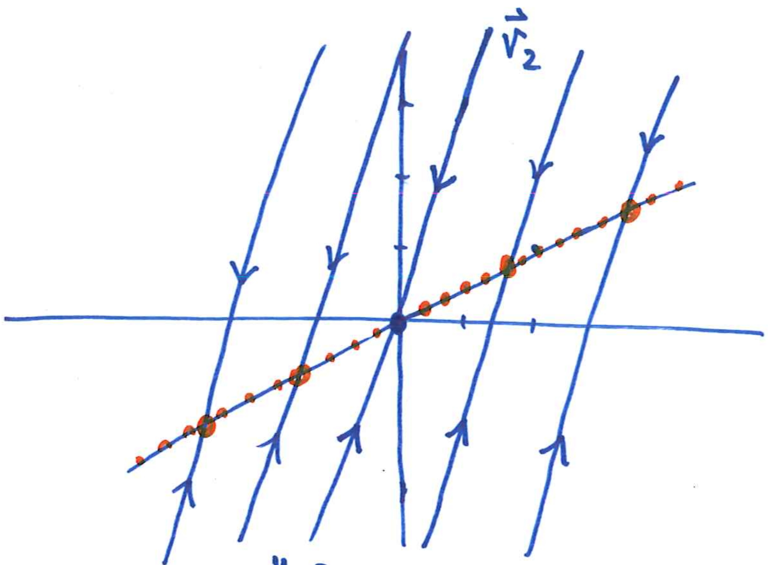
All critical points of the system lie on $\vec{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$r_2 = -5 \Rightarrow (A+5I)v = 0 \Rightarrow \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 3v_1 = v_2$$

$$\Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{V}_2$$

$$\vec{X}_2(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-5t} = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} \begin{matrix} = x \\ = y \end{matrix}$$

$\hookrightarrow$ $y = 3x$



$$\vec{X}(t) = c_1\vec{X}_1 + c_2\vec{X}_2 = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-5t}$$

$$= \begin{pmatrix} 2c_1 + c_2 e^{-5t} \\ c_1 + 3\underbrace{c_2 e^{-5t}} \end{pmatrix} \begin{matrix} \to x \\ \to y \end{matrix} \Rightarrow c_2 e^{-5t} = \underbrace{x - 2c_1}$$

$$y = c_1 + 3(x - 2c_1) = 3x + \cdots$$

$\hookrightarrow$ all other solutions are parallel to $(y=3x)$ which is the line through $\vec{V}_2$ and each line converges to a critical point.

Example 2 . $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\det \begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} = r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$$

$\Rightarrow r = 1$ repeated eigval

$$r = 1 \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow v_1 = 2v_2 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \vec{v}$$

$$\vec{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t = \begin{pmatrix} 2e^t \\ e^t \end{pmatrix}$$

$\vec{x}_2(t) = (t\vec{v} + \vec{w}) e^t$ where $\vec{w}$ solves the

equation: $(A - rI)\vec{w} = \vec{v}$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right) \div 2 \quad \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow w_1 - 2w_2 = 1 \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{w}$$

So

$$\vec{x}_2(t) = \left(t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t = \begin{pmatrix} 2t+1 \\ t \end{pmatrix} e^t$$

$$\Rightarrow X(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left(t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^t$$

$\rightarrow$ improper node unstable

$$= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2t+1 \\ \end{pmatrix} e^t$$

Today: Fundamental matrix and Wronskian.

$$\underset{n\times 1}{\vec{X}'} = \underset{n\times n}{A} \underset{n\times 1}{X}$$

Let's assume $\vec{X}_1(t), \vec{X}_2(t), \dots, \vec{X}_n(t)$ are solutions to the system.

Then if $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent

then $\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + \dots + c_n\vec{X}_n(t)$ is also

This covers all possible solutions

a solution.

<u>Definition</u>: $\vec{X}_1, \vec{X}_2, \dots, X_n$ are called linearly independent on the interval I if $c_1\vec{X}_1 + \dots + c_n\vec{X}_n = 0$

then $c_1 = c_2 = \dots = c_n = 0$ for all t in I.

<u>Definition</u>:

Fundamental set of solutions: A set that contains all linearly independent solutions of the system

$$\{\vec{X}_1(t), \vec{X}_2(t), \dots, \vec{X}_n(t)\}$$

Fundamental matrix of system $\vec{x}' = A\vec{x}$

$$\mathbb{X}(t) = \left(\vec{x}_1(t) \;\Big|\; \vec{x}_2(t) \;\Big|\; \cdots \;\Big|\; \vec{x}_n(t)\right)$$

a matrix whose columns are linearly independent solutions.

Note: If $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$ in order to have linear independent $x_1, \dots, x_n$

$$\left(\vec{x}_1 \;\Big|\; \vec{x}_2 \;\Big|\; \cdots \;\Big|\; \vec{x}_n\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0$$

must have a unique solution so we require $\det \mathbb{X}(t) \neq 0$ for all t

Fact: $\underbrace{\text{then}}_{\text{if } \mathbb{X}(t) \text{ is the fundamental matrix}}$ $\det \mathbb{X}(t)$ is either $\underline{\text{always } 0}$ linear dependent $\vec{x}_i$

or $\underline{\text{NEVER } 0}$. linear independent $\vec{x}_i$

Proof: For 2×2, linear, autonomous, homogeneous:

$$\vec{x}' = A\vec{x} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Say $\vec{X}_1(t)$ and $\vec{X}_2(t)$ are two solutions so we have

$$\vec{X}_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \xrightarrow{\vec{x}_1' = A\vec{X}_1} \begin{cases} ax_1 + by_1 = x_1' \\ cx_1 + dy_1 = y_1' \end{cases} \quad \text{I}$$

$$\vec{X}_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \xrightarrow{\vec{X}_2' = A\vec{X}_2} \begin{cases} ax_2 + by_2 = x_2' \\ cx_2 + dy_2 = y_2' \end{cases} \quad \text{II}$$

$$\mathbb{X}(t) = \left(\vec{X}_1(t) \,\vdots\, \vec{X}_2(t) \right) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

Claim : $\det \mathbb{X}(t) = 0$ always or $\neq 0$ ~~NEVER~~ Always

$$\det \mathbb{X}(t) = x_1 y_2 - x_2 y_1$$

$$\frac{d}{dt}\left(\det \mathbb{X}(t)\right) = x_1' y_2 + x_1 y_2' - x_2' y_1 - x_2 y_1'$$

$= $ replace x_1', y_2', x_2', y_1' from I and II

bunch of algebra $\vdots$

$$= (a+d)\, x_1 y_2 - (a+d)\, x_2 y_1$$

$$= (a+d)(x_1 y_2 - x_2 y_1)$$

$$= (a+d) \det \mathbb{X}(t)$$

solve for $\det \mathbb{X}(t)$:

$$\det \mathbf{X}(t) = K e^{(a+d)t}$$

if $K = 0$ then $\det \mathbf{X}(t) = 0$ for all t

if $K \neq 0$ then $\det \mathbf{X}(t) \neq 0$ for all t. $\blacksquare$

This implies that x_1 and x_2 are either always linearly independent or NEVER linearly indep.

<u>Remark</u>: $\det \mathbf{X}(t)$ is called Wronskian of $\vec{x}_1$ and $\vec{x}_2$.

Notation: $W[\vec{x}_1, \vec{x}_2]$

$\Rightarrow W[\vec{x}_1, \vec{x}_2] = 0$ <u>always</u> or <u>NEVER 0</u>.

$\Downarrow$ linear dep

linear indep

Example 2:

$$\begin{array}{cc} & x_1 \qquad\qquad x_2 \\ \mathbf{X}(t) = & \begin{pmatrix} 2e^t & (2t+1)e^t \\ e^t & te^t \end{pmatrix} \end{array}$$

Fundamental matrix

$$\det \mathbb{X}(t) = 2\cancel{t}e^{2t} - 2\cancel{t}e^{2t} - e^{2t}$$

$$= -e^{2t} \neq 0$$

Ex 1 :

$$\mathbb{X}(t) = \begin{pmatrix} 2 & e^{-5t} \\ 1 & 3e^{-5t} \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad x_1 \qquad\quad x_2$$

$$W[x_1, x_2] = 6e^{-5t} - e^{-5t} = 5e^{-5t} \neq 0$$

---

matrix

NEXT Class: $e^{At} = ?$