Reminder :

Oct 17
Lecture 18

- WebWork 3 : due this Friday , Oct 19
- MATLAB Tutorial :

today 10 – 11 am ~~X~~ } LSK 121
Thursday , Oct 18 1-2 pm } LSK 121
(tomorrow)

~~Recall from last class :~~

$$\underset{n\times 1}{\vec{x}'} = \underset{n\times n}{A(t)}\underset{n\times 1}{\vec{x}}$$ linear , homogeneous

if $\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)$ are linearly independent solutions of the system :

$$\mathbb{X}(t) = \left( \vec{x}_1(t) \,\vdots\, \vec{x}_2(t) \,\vdots\, \cdots \,\vdots\, \vec{x}_n(t) \right)$$

$\det \mathbb{X}(t) = W[x_1, x_2, \dots, x_n]$ ⇝ Wronskian

<u>Fact</u> : $W[x_1, \dots, x_n]$ is always zero on an interval $I : \alpha < t < \beta$ or NEVER $0$ on $I$.

This Fact is useful :

Given $x_1, x_2, \dots, x_n$ ; we only need to examine $W[x_1, x_2, \dots, x_n]$ at one point for which the computation is easy .

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If $\vec{x}_1, \dots, \vec{x}_n$ are linearly indep then

General solution of $\vec{x}'(t) = A\,\vec{x}(t)$ , $\vec{x}(0) = \vec{x}_0$ is

$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \dots + c_n \vec{x}_n(t)$ then $\vec{x}(0) = c_1 \vec{x}_1(0) + \dots + c_n \vec{x}_n(0) = \vec{x}_0$

$$\mathbb{X}(t) = \left( \vec{x}_1 \;\vdots\; \vec{x}_2 \;\vdots\; \cdots \;\vdots\; \vec{x}_n \right)$$

in matrix language:

$$\vec{x}(t) = \mathbb{X}(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{then} \quad \vec{x}(0) = \mathbb{X}(0)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}_0$$

$\det \mathbb{X}(0) \neq 0$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathbb{X}^{-1}(0)\,\vec{x}_0$$

$$\boxed{\vec{x}(t) = \mathbb{X}(t)\left(\mathbb{X}(0)\right)^{-1}\vec{x}_0}$$

We didn't find $\vec{x}(t)$ this way, but in higher dimensional systems this approach is useful.

Solution in matrix language

Recall:

$$x' = ax \longrightarrow x(t) = Ce^{at} \xrightarrow[C = x_0]{x(0)=x_0} x(t) = x_0 e^{at}$$

$$\vec{x}' = \underbrace{A}_{\text{Constant matrix}} \vec{x} \longrightarrow X(t) = C\boxed{e^{At}}$$

How to interpret? → Matrix Exponential?

?

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots$$

We know the series is converging to $e^{at}$ for all $t$.

Similarly:

matrix ←

$$e^{At} \stackrel{\text{def}}{=} I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

each term is an $n \times n$ matrix

each $\underset{\text{element}}{\underline{\text{term}}}$ in the sum matrix converges, take the limit of convergence and call it $e^{tA}$.

$$\frac{d}{dt}(e^{tA}) = 0 + A + \frac{A^2}{2!} \cdot 2t + \frac{A^3}{3!} 3t^2 + \cdots$$

$$= A\underbrace{\left( I + At + \frac{A^2}{2!}t^2 + \cdots \right)}$$

$$\frac{d}{dt}\left(e^{At}\right) = A e^{At} \rightsquigarrow \text{Convention: } tA$$

1D number

A matrix

$$\frac{d}{dt}\left(e^{tA}\right) = A e^{tA} \rightsquigarrow \text{This implies } e^{tA} \text{ solves}$$

$$\frac{d}{dt}\vec{X} = A\vec{X} \quad (\vec{X}' = A\vec{X})$$

So $e^{tA}$ is a solution to the system.

How to compute $e^{tA}$ ?

(1) A is diagonal : $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Observe that $A^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \cdots A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$A_{2\times2}$ :

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ta & 0 \\ 0 & tb \end{pmatrix} + \begin{pmatrix} \frac{t^2a^2}{2!} & 0 \\ 0 & \frac{t^2b^2}{2!} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + ta + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \cdots & 0 \\ 0 & 1 + tb + \frac{t^2b^2}{2!} + \frac{t^3b^3}{3!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$$

$$\boxed{\text{So } e^{tA} = e^{t\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}}$$

(2) $A_{n\times n}$ is diagonalizable

This is possible when $A$ has $n$ linearly independent eigvec $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ corresponding to distict eigval $r_1, r_2, \dots, r_n$.

Matrix of eigvec $E = \left( \vec{v}_1 \,\vdots\, \vec{v}_2 \,\vdots\, \cdots \,\vdots\, \vec{v}_n \right) \xrightarrow[\text{lin indep}]{\vec{v}_i} \det E \neq 0$

It's easy to check that:

$$E^{-1} A E = \underbrace{\begin{pmatrix} r_1 & & 0 \\ & r_2 & \\ 0 & & \ddots \; r_n \end{pmatrix}} = D$$

then $A = E D E^{-1}$

Observe that:

$A^2 = E D \underbrace{E^{-1} E} D E^{-1} = E D^2 E^{-1}$

$\vdots$

$A^n = E D^n E^{-1}$

**Example.** $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ; Find $e^{tA}$

$$\det\begin{pmatrix} 1-r & 2 \\ 2 & 1-r \end{pmatrix} = 0 \Rightarrow r^2 - 2r - 3 = 0$$

$$r_1 = 3 \Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ A is diagonalizable.

$$r_2 = -1 \Rightarrow \vec{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$E = \begin{pmatrix} \underbrace{\begin{matrix}1\\1\end{matrix}}_{\vec{V}_1} & \underbrace{\begin{matrix}-1\\1\end{matrix}}_{\vec{V}_2} \end{pmatrix} \xrightarrow{\det E = 2} E^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

* Recall: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A = ad - bc$

$A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- Verify that $A = EDE^{-1}$

$$e^{tA} = e^{t(EDE^{-1})} = I + tEDE^{-1} + \frac{t^2}{2!}(EDE^{-1})^2 + \frac{t^3}{3!}(EDE^{-1})^3 + \cdots$$

$$= \underbrace{I}_{EIE^{-1}} + tEDE^{-1} + \frac{t^2}{2!}ED^2E^{-1} + \frac{t^3}{3!}ED^3E^{-1} + \cdots$$

$$= E\left(I + tD + \frac{t^2}{2!}D^2 + \frac{t^3}{3!}D^3 + \cdots\right)E^{-1}$$

$$= E\, e^{tD} E^{-1}$$

So

$$\boxed{e^{tA} = e^{tEDE^{-1}} = E\, e^{tD}\, E^{-1}}$$

D is diagonal and from case (1) we know how to compute $e^{tD}$

Ex. (cont'd)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad E^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e^{tA} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{t\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(1)

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

You do it ⋮

$$= \frac{1}{2}\begin{pmatrix} e^{3t} - e^{-t} & e^{3t} + e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{pmatrix}$$

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$$\vec{x}' = A\vec{x} \Rightarrow \vec{x}(t) = C e^{tA}$$

$$x(0) = x_0 \quad x(0) = C e^{0A} = C I = x_0 \Rightarrow C = x_0$$

$$\Rightarrow \vec{x}(t) = e^{tA} x_0$$

remains unchanged for different initial cond'n.

This is useful as it makes computations easy for different initial conditions.