Oct 19
Lecture 19 ☺

Last day:

$X' = A(t)X$ $\quad$ 1^{st} order linear

A is constant:

Exponential Matrix

$$e^{tA} \overset{def}{=} I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$$

$$\frac{d}{dt}(e^{tA}) = (e^{tA})' = A e^{tA} \Rightarrow e^{tA} \text{ is a solution to } x' = AX$$

in general: $\vec{X}(t) = e^{tA} \cdot \vec{C}$ is a solution to $x' = AX$.

computation of e^{tA}:

(1) A diagonal $\quad A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$e^{tA} = e^{t\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} e^{ta} & 0 \\ 0 & e^{tb} \end{pmatrix}$$

(2) $A_{n\times n}$ diagonalizable

when A has n linearly indep eig vec $\vec{V}_1, \vec{V}_2, \dots, \vec{V}_n$ corresponding to eigval $r_1, r_2, \dots, r_n$

$$E = \left(\vec{V}_1 \,\vdots\, \vec{V}_2 \,\vdots\, \cdots \,\vdots\, \vec{V}_n\right)$$

$$A = EDE^{-1} \quad \text{where} \quad D = \begin{pmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_n \end{pmatrix}$$

$$e^{tA} = e^{t(EDE^{-1})} = E\, e^{tD}\, E^{-1}$$

$$\vec{x}\,'(t) = A\vec{x} \rightsquigarrow \vec{x}(t) = e^{tA}\vec{C}$$

$$\xrightarrow{\vec{x}(0)=\vec{x}_0} \quad \vec{C} = \vec{x}_0$$

One way of representing the solution: $\rightarrow$ $\boxed{\vec{x}(t) = e^{tA}\cdot\vec{x}_0}$ I

2nd way: (usual way)

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

$$= \left(\vec{x}_1 \,\vdots\, \vec{x}_2 \,\vdots\, \cdots \,\vdots\, \vec{x}_n\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= X(t)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$\xrightarrow{\vec{x}(0)=\vec{x}_0} \quad \vec{x}_0 = X(0)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = X^{-1}(0)\,\vec{x}_0$$

then $\boxed{\vec{x}(t) = X(t)\,X^{-1}(0)\,\vec{x}_0}$ II

Compare I and II :

$$\boxed{e^{tA} = \mathbb{X}(t)\,\mathbb{X}^{-1}(0)}$$

another way of defining e^{tA}.

Use this definition to find e^{tA} when:

3. A is NOT diagonalizable :

in 2D : A has repeated eigvalues with only one eigvec

A : defective

<u>Example</u> : If $A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$ find e^{tA}.

$r = -1 \rightsquigarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightsquigarrow \vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

$\vec{w}$ generalized eigvec

$(A - rI)\vec{w} = \vec{v}$

$\rightsquigarrow \vec{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rightarrow \vec{x}_2(t) = \left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] e^{-t}$

$$\Rightarrow \mathbb{X}(t) = \left(\vec{x}_1(t) \;\middle|\; \vec{x}_2(t) \right) = \begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (-t-1)e^{-t} \end{pmatrix}$$

Find $\mathbb{X}(0)$, invert it to get $\mathbb{X}^{-1}(0)$ and find

$$\mathbb{X}(t)\,\mathbb{X}^{-1}(0) = e^{tA} = \begin{pmatrix} e^{t} - te^{t} & -te^{t} \\ te^{t} & (t+1)e^{t} \end{pmatrix}$$

Nonhomogeneous linear systems of ODEs.

forced

$$\circledast \quad \vec{X}'(t) = A(t)\vec{X}(t) + \underbrace{\vec{G}(t)}_{\text{force function}}$$

in 2x2 case:

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \qquad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$\begin{cases} x'(t) = a(t)\,x(t) + b(t)\,y(t) + g_1(t) \\ y'(t) = c(t)\,x(t) + d(t)\,y(t) + g_2(t) \end{cases}$$

General Solutions of $(\star)$:

$$\vec{X}(t) = \underset{\text{sol'n}}{\text{Homogeneous}} + \underset{\text{sol'n}}{\text{Particular}}$$

$$= \vec{X}_H + \vec{X}_p$$

$\vec{X}_H$: solves the Homogeneous system corresponding to $(\star)$ i.e.

$$\vec{X}_H' = A(t)\,\vec{X}_H$$

involves $c_1, c_2, \ldots, c_n$ $\quad X_H(t) = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$

$\vec{X}_p$: any particular solution that solves the non-hom equation i.e. $\vec{X}_p' = A\vec{X}_p + \vec{G}$

NO constants

Why is $\vec{x}(t) = \vec{x}_H + \vec{x}_p$ a solution ?

plug into $(\star)$: $\vec{x}\,' = A\vec{x} + \vec{G}$

LHS: $\vec{x}\,' = (\vec{x}_H + \vec{x}_p)' = \underbrace{\vec{x}_H'} + \underbrace{\vec{x}_p'}$

$$= \underbrace{A\vec{x}_H} + \underbrace{A\vec{x}_p + \vec{G}}$$

$$= A(\underbrace{\vec{x}_H + \vec{x}_p}) + \vec{G}$$

$$= A\vec{x} + \vec{G} \quad \checkmark$$

: RHS

Compare with 1D case :

$$y' + 2y = 2e^{t/3}$$

integrating factor $r(t) = e^{2t}$

$$y(t) = e^{-2t}\left(\int e^{2t}\cdot 2e^{t/3} + C\right)$$

$$= \vdots$$

$$= \underbrace{\frac{6}{7}e^{t/3}}_{x_p} + \underbrace{Ce^{-2t}}_{x_H}$$

$x_H = Ce^{-2t}$ solves the hom eqt: $y' + 2y = 0$

$$-2Ce^{-2t} + 2Ce^{-2t} = 0 \checkmark$$

$x_p = \frac{6}{7} e^{t/3}$ solves

$$y' + 2y = 2e^{t/3}$$

$$\text{LHS}: \frac{6}{21} e^{t/3} + \frac{12}{7} e^{t/3} = \frac{42}{21} e^{t/3} = 2e^{t/3} \checkmark : \text{RHS}$$

$x(t) = x_p(t) + x_H(t)$ is infact a solution.

$\boxed{\text{Question: How to find } x_p \text{ for systems ?}}$

Let is make an amology for the systems:

$$x' = a(t)\, x + g(t)$$

$r(t)$: integ factor

$$x(t) = (r(t))^{-1} \left(\int r(t) \cdot g(t)\, dt + C \right)$$

$$= \underbrace{r(t)^{-1} \int r(t)\, g(t)\, dt}_{x_p} + \underbrace{C (r(t))^{-1}}_{x_H}$$

$$\vec{x}' = A(t)\, \vec{x} + \vec{G}(t)$$

solution to homogeneous equation: $\vec{x}' = AX$ is

of the form: $x(t) = \mathbb{X}(t) \cdot \vec{C}$

$= c_1 X_1 + c_2 X_2 + \dots + c_n X_n$

Observe that:

- $\mathbb{X}^{-1}(t)$ plays the role of

  integrating factor $r(t)$

Also, 1D case suggests that

$$\vec{x}_p(t) = \mathbb{X}(t) \underbrace{\int \mathbb{X}^{-1}(t)\, G(t)\, dt}$$

$$= \mathbb{X}(t)\, U(t)$$

Note that:

$$\begin{cases} \bullet\ U'(t) = \mathbb{X}^{-1}(t)\, G(t) \\ \bullet\ A\,\mathbb{X}(t) = \mathbb{X}'(t) \end{cases}$$

($\mathbb{X}(t)$: contains all the solutions in each of its columns.)

plug $\vec{x}_p$ into the equation: $\vec{x}' = A\vec{x} + \vec{G}$

$$x_p' = \mathbb{X}'U + \mathbb{X}U' = A\underbrace{\mathbb{X}U}_{x_p} + \underbrace{\mathbb{X}\mathbb{X}^{-1}}_{I} G$$

$$= A\vec{x}_p + \vec{G}\ \checkmark$$

so $\vec{x}_p$ is in fact a particular solution to non-hom system

$\Rightarrow$ Then the general sol'n is

$$\boxed{\begin{aligned} \vec{x}(t) &= \vec{x}_H(t) + \vec{x}_p(t) \\ &= \underbrace{\mathbb{X}(t)\, C}_{\text{already knew}} + \underbrace{\mathbb{X}(t) \int \mathbb{X}^{-1}(t)\, G(t)\, dt}_{\text{matrix algebra with fundamental matrix}} \end{aligned}}$$

This solution is valid for $\boxed{\text{any}}$ linear system.

* Note: Apply initial condition to find $\vec{C}$ at this step and not earlier.

Ex. Solve:

$$\vec{x}' = \underbrace{\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}}_{\text{Homogeneous part}} \vec{x} + \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} \rightarrow \text{force term}$$

x_H:

$$r_1 = -1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

$$r_2 = -6 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

$$\mathbb{X}(t) = \begin{array}{c} \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \\ \quad X_1 \qquad X_2 \end{array}$$

$$\vec{x}_H = \mathbb{X}(t)\,\vec{c} = c_1 X_1 + c_2 X_2 = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

Finding $\mathbb{X}^{-1}(t)$:

we want to compute

$$\vec{x}_p = \mathbb{X}(t) \int \mathbb{X}^{-1}(t) G(t)\, dt$$

$$\det \mathbb{X}(t) = -5e^{-7t}$$

$$\mathbb{X}^{-1}(t) = \frac{1}{-5e^{-7t}} \begin{pmatrix} -2e^{-6t} & -e^{-6t} \\ -e^{-t} & 2e^{-t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{t} & e^{t} \\ e^{6t} & -2e^{6t} \end{pmatrix}$$

$$\vec{x}_p = X \int \frac{1}{5} \begin{pmatrix} 2e^{t} & e^{t} \\ e^{6t} & -2e^{6t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} dt$$

$$= \frac{1}{5} X \int \begin{pmatrix} 2e^{-t} \\ e^{4t} \end{pmatrix} dt$$

$$= \frac{1}{5} X \begin{pmatrix} -2e^{-t} \\ \frac{1}{4} e^{4t} \end{pmatrix}$$

$$\vec{x}_p = \frac{1}{5} \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} -2e^{-t} \\ \frac{1}{4} e^{4t} \end{pmatrix} = e^{-2t} \begin{pmatrix} -3/4 \\ -1/2 \end{pmatrix}$$

General solution:

$$\boxed{\begin{aligned} \vec{x} &= \vec{x}_H + \vec{x}_p \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} + e^{-2t} \begin{pmatrix} -3/4 \\ -1/2 \end{pmatrix} \end{aligned}}$$

Given the initial condition $\vec{x}(0) = \vec{x}_0$

you solve for c_1 and c_2 now.