Lecture 20
Oct 22

Last Class :

Non-homogeneous (forced) systems of $\underbrace{\text{ODEs}}$ linear

(*) $\vec{x}\,'(t) = A(t)\,\vec{x}(t) + \underbrace{\vec{G}(t)}_{\text{force function}}$

General Solution :

$$\vec{x}(t) = \underbrace{\vec{x}_H(t)}_{\text{Solution of the Homogeneous Equation}} + \underbrace{\vec{x}_{p}(t)}_{\text{One particular solution that solves the non-hom system}}$$

Solution of the Homogeneous Equation
$\vec{x}_H' = A(t)\,\vec{x}_H$

already knew how to find $\vec{x}_H$:
find $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ and

$$\vec{x}_H = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$$

$$= \left(\vec{x}_1 \,\vdots\, \vec{x}_2 \,\vdots\, \dots \,\vdots\, \vec{x}_n\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \underbrace{\mathbb{X}(t)}_{\text{Fundamental matrix}}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Fundamental matrix

One particular solution that solves the non-hom system :
$\vec{x}_p' = A(t)\,\vec{x}_p + \vec{G}(t)$

New : Comparing with the 1D linear equation
$x' = a(t)x + g(t)$

We found that :

$$\boxed{\vec{x}_p = \mathbb{X}(t)\int \mathbb{X}^{-1}(t)\,G(t)\,dt}$$

Method of Variation of Parameters to find $\vec{x}_p$

Therefore :

General solution of (*) :

$$x(t) = \mathbb{X}(t)\,C + \mathbb{X}(t)\int \mathbb{X}^{-1}(t)\,G(t)\,dt \,.$$

solves any linear system

<u>Exercise</u> : Show that the general solution to the forced system (*) when A is a constant matrix can be written as

$$x(t) = e^{At}\vec{x}_0 + e^{At}\int e^{-At}\,G(t)\,dt$$

where $\vec{x}(0) = \vec{x}_0$.

We did one example

Ex 1. $\vec{x}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$

where A is a constant matrix.

Now

Ex 2. $\vec{x}' = \begin{pmatrix} 1 & 0 \\ -\frac{4}{t^2} & -\frac{2}{t} \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}$, $\vec{x}(1) = \begin{pmatrix} e \\ -2e \end{pmatrix}$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

time dependent A

eigval & eigvec not working here

$$\Rightarrow \begin{cases} x' = x \\ y' = -\frac{4}{t^2} x - \frac{2}{t} y + t \end{cases}$$

solve this for x:

system can be uncoupled and solved separately for each unknown.

$x' = x \Rightarrow \boxed{x(t) = c_1 e^t}$

plug into 2nd

$$y' = -\frac{4c_1}{t^2} e^t - \frac{2}{t} y$$

$$\longrightarrow \quad y' + \frac{2}{t} y = \frac{-4}{t^2} c_1 e^t$$

1st order 1D linear equation

Chapter 1 → find the integrating factor and solve:

$$\boxed{y(t) = \frac{-4c_1}{t^2} e^t + \frac{c_2}{t^2}}$$

$$X(t) = \begin{pmatrix} c_1 e^t \\ -\frac{4c_1}{t^2} e^t + \frac{c_2}{t^2} \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} e^t \\ -\frac{4}{t^2} e^t \end{pmatrix}}_{X_1} + c_2 \overbrace{\begin{pmatrix} 0 \\ \frac{1}{t^2} \end{pmatrix}}^{X_2}$$

$$\mathbb{X}(t) = \begin{pmatrix} e^t & 0 \\ \frac{-4e^t}{t^2} & \frac{1}{t^2} \end{pmatrix}$$

You verify that $W[\vec{x}_1, \vec{x}_2] \neq 0$

$\vec{x}_H$ found , Now we find $\vec{x}_p$:

$$\vec{x}_p(t) = \mathbb{X}(t) \int \mathbb{X}^{-1}(t)\, G(t)\, dt$$

$* \det \mathbb{X}(t) = \frac{e^t}{t^2}$

$\mathbb{X}^{-1}(t) = \frac{t^2}{e^t}\begin{pmatrix} & - \\ - & \end{pmatrix}$

$$= \mathbb{X}(t) \int \frac{t^2}{e^t} \begin{pmatrix} \frac{1}{t^2} & 0 \\ \frac{4e^t}{t^2} & e^t \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} dt$$

$$= \mathbb{X}(t) \int \begin{pmatrix} e^t & 0 \\ 4 & t^2 \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} dt$$

$$= \mathbb{X}(t) \int \begin{pmatrix} 0 \\ t^3 \end{pmatrix} dt$$

$$= \begin{pmatrix} e^t & 0 \\ \frac{-4e^t}{t^2} & \frac{1}{t^2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{4}t^4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}t^2 \end{pmatrix}$$

$$\Rightarrow \vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

$$= c_1 \begin{pmatrix} e^t \\ \frac{-4}{t^2}e^t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \frac{1}{t^2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{4}t^2 \end{pmatrix}$$

$$\vec{x}(1) = \begin{pmatrix} e \\ -2e \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} e \\ -2e \end{pmatrix} = \begin{pmatrix} c_1 e \\ -4c_1 e + C_2 + \frac{1}{4} \end{pmatrix}$$

$$\Rightarrow \quad c_1 = 1$$

$$-2e = -4e + c_2 + \frac{1}{4} \Rightarrow c_2 = 2e - \frac{1}{4}.$$

Exercise : Consider the system

$$\vec{X}'_{n\times 1} = A_{n\times n} \vec{X}_{n\times 1} = G_{n\times n}(t)$$

Constant matrix

(a) Suppose A is diagonalizable i.e. $A = EDE^{-1}$

Use this fact to uncouple the system into n 1D linear ODEs and find the general solution.

(b) Suppose A is defective i.e. repeated eigvalues

How would matrix D look like? still diagonal?

How can we uncouple the system?

Another Method to find a particular solution $\vec{x}_p$ is

Undetermined Coefficients.

Example : $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$ $(\star)$

First solve the homogeneous system to find x_H :

$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} \Rightarrow r^2 + 4r + 3 = 0$

$r_1 = -1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_2 = -3 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \boxed{\vec{x}_H = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}}$

General Solution of $(\star)$ is :

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

How to find $\vec{x}_p$? Check the force function and guess $\vec{x}_p$ based on the family of functions appearing in $G(t)$:

$G(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} \rightsquigarrow$ Guess : $\vec{x}_p = \vec{a}\, t e^{-t} + \vec{b} e^{-t} + \vec{c}\, t + \vec{d}$

all possible combinations of t & e^{-t} :

Assume $\vec{x}_p$ is a particular sol'n so it satisfies the equation

$$\vec{x}_p' = A \vec{x}_p + \vec{G}$$

and by equating the two sides, coefficients are found.

LHS: $\vec{x}_p' = \vec{a}e^{-t} - \vec{a}te^{-t} - \vec{b}e^{-t} + \boxed{\vec{c}}$

RHS: $A\vec{x}_p + \vec{G} = A\vec{a}\,te^{-t} + A\vec{b}\,e^{-t} + A\vec{c}\,t + \boxed{A\vec{d}} + \begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3t \end{pmatrix}$

$\begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t}$ $\quad \begin{pmatrix} 0 \\ 3 \end{pmatrix}t$

LHS = RHS

$\Rightarrow \quad A\vec{a} = -\vec{a} \xrightarrow{\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{a}$

$A\vec{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \vec{a} - \vec{b}$

$A\vec{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 0$

$A\vec{d} = \vec{c}$

$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -b_1 + b_2 \\ b_1 - b_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \Rightarrow \vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{d} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}$

So $\boxed{\vec{x}_p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t + \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}}$

$\Longrightarrow \vec{X}(t) = \vec{x}_H + \vec{x}_p$ Copy them :

Some example of force function:

$$G(t) = \begin{pmatrix} t^2 \\ 0 \end{pmatrix} \Rightarrow \vec{X}_p(t) = \vec{a}\, t^2 + \vec{b}\, t + \vec{c}$$

$$G(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix} \Rightarrow \vec{X}_p(t) = \vec{a}\, t \sin t + \vec{b}\, t \cos t + \vec{c} \sin t + \vec{d} \cos t + \vec{e}\, t + \vec{f}$$

Chapter 2 :

Recall : $y' = f(t, y)$

2^{nd} order ODE :

y''

general form: $y'' = f(t, y, y')$

$$\frac{d^2 y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Linear 2^{nd} order ODE :

$$y'' + p(t)\, y' + q(t)\, y = g(t)$$

in y

y'', y', y

e^y X

y^2 X

$e^{y'}$ X

$\sin y''$ X

if $g(t) = 0 \Rightarrow$ Homogeneous.

2^{nd} order ODE $\longleftrightarrow$ 1^{st} order system

From a 2^{nd} order 1D ode to a 1^{st} order system

$$y'' + p(t)y' + q(t)y = g(t)$$

Define new variables $\begin{cases} y = u \Rightarrow u' = y' = v \Rightarrow \mathrm{I}\ \boxed{u' = v} \\ y' = v \Rightarrow \boxed{v' = y'' = -p(t)y' - q(t)y + g(t)} \end{cases}$

$$\boxed{v' = -p(t)v - q(t)u + g(t)} \quad \mathrm{II}$$

Write I and II as a system:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \overrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} + \vec{G}(t)$$

Go from a system 1^{st} order to 2^{nd} order ODE

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \longrightarrow y'' = cx' + dy'$$

$$= c(ax + by) + dy'$$

$$x = \frac{y' - dy}{c} \qquad y'' = c\left(a\frac{y'-dy}{c} + by\right) + dy'$$

$$\Rightarrow y'' - (a+d)y' + (ad - bc)y = 0$$