MATLAB Center:

Hours: Wednesday, Oct 24 : 10 am - 1 pm

Thursday, Oct 25 : 11 am - 2 pm

Friday, Oct 26 : 11 am - 1 pm

Jordan's Tutorials :

(Section 102 instructor)

Friday, Oct 26 : 1 - 2 pm

Wednesday, Oct 31 : 1 - 2 pm

Thursday, Nov 1 : 1 - 2 pm

Oct 24

Lec 21

2nd order ODEs:

$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = y_0 \\ y'(t_0) = \bar{y}_0 \end{cases}$$

2nd-order IVP

y_0 and $\bar{y}_0$ are constant numbers.

2nd order linear ODE

$$y'' + p(t)\, y' + q(t)\, y = g(t)$$

if $g(t) = 0$ $\rightsquigarrow$ 2nd order linear homogeneous ODE

2^{nd} order linear homogeneous with constant coefficient

$$y'' + by' + cy = 0$$

b, c: constants

(if $ay'' + by' + cy = 0$ divide by a)

Theorem (Existence & Uniqueness of the linear case)

$$y'' + p(t)y' + q(t)y = g(t) \quad , \quad y(t_0) = y_0 \ , \ y'(t_0) = \bar{y}_0$$

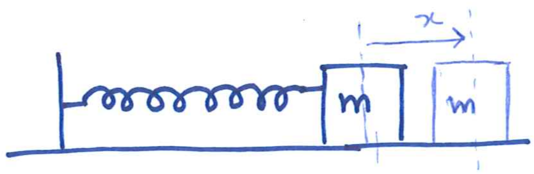
If $p(t)$, $q(t)$ and $g(t)$ are continuous on some interval I containing t_0, then the equation has exactly one solution, $y = \phi(t)$, and $\phi(t)$ is defined on the same interval I.

i.e. Interval of existence = Interval of continuity $= I$.

Let's see some physical applications of the 2^{nd} order ODEs before doing mathematical computation to find the general solution.

Application of 2^{nd} order ODE :

(1) Vibrating Springs (Mass - Spring system)



(vertical or horizontal, both work.)

A spring attached to an object with mass m . Stretch or Compress the spring x units in length

Hooke's law : The spring exerts a force proportional to the displacement x and in the opposite direction of the displ. x $\rightsquigarrow$ Restoring force

Newton's 2^{nd} law : $F = -kx$

spring constant $(k > 0)$

$$ma = -kx$$

$$a = \frac{d^2x}{dt^2} \Rightarrow m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \boxed{x'' + \frac{k}{m}x = 0}$$

Undamped equation

$\rightarrow$ 2^{nd} order linear

All external forces such as friction & air resistance are neglected $\rightsquigarrow$ system called : Undamped motion

$\hookrightarrow$ Oscillation continuous forever !

Let's consider resisting forces (friction or air resistance)

↳ Oscillation dies off slowly :

↳ This effect is called damping.

Assumption: Damping force is proportional to the velocity of the object & in the opposite direction of V.

$$\text{damping force} = -cV = -c\frac{dx}{dt}$$

$$F = -kx - c\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$$

→ 2nd order linear hom. damped.

Apply some additional external force to the system F:

$$\boxed{\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = F(t)}$$

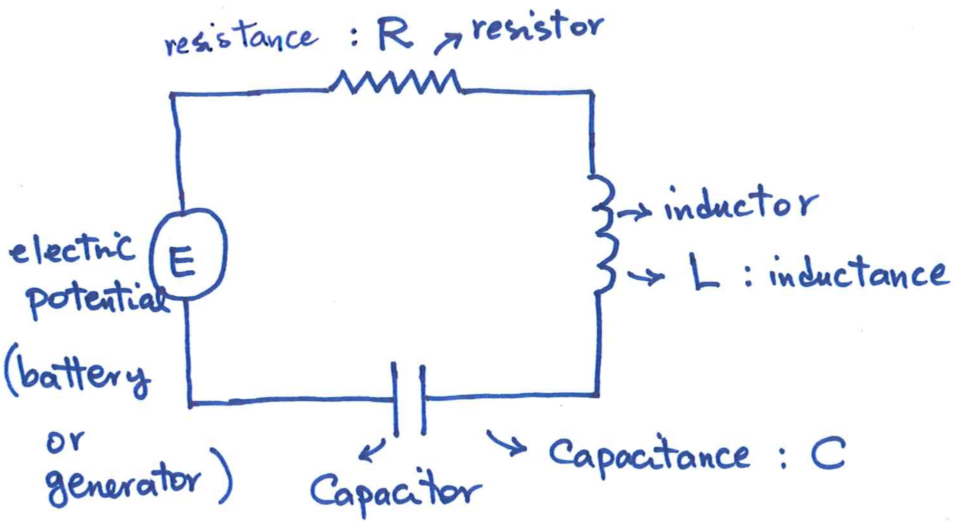
→ 2nd order linear non-hom damped

↳ forced 2nd order ODE

$$\begin{cases} x(0) = x_0 \\ V(0) = V_0 \end{cases}$$

(2) The RLC Circuit (series)

An electric circuit containing a resistor, an inductor and a capacitor.



$I(t)$: the current in the circuit at time t

$q(t)$: the charge on the capacitor at time t.

Facts :

- $I(t) = \dfrac{dq(t)}{dt} \rightarrow \dfrac{dI}{dt} = \dfrac{d^2q}{dt^2}$
- Voltage drops across resistor, inductor and capacitor.

voltage across the inductor $\swarrow$ $$E_L = L\frac{dI}{dt}$$

voltage across the resistor $\swarrow$ $$E_R = RI$$

voltage across the capacitor $\swarrow$ $$E_C = \frac{1}{C}q(t)$$

Kirchhoff's 2^{nd} law : Sum of the voltage drop

= sum of the supplied voltage

$$\Rightarrow \quad E_L + E_R + E_C = E(t)$$

$$\Rightarrow \quad L\frac{dI}{dt} + RI + \frac{1}{C}q(t) = E(t)$$

$$\Rightarrow \quad \boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q(t) = E(t)}$$

2^{nd} order linear ODE

$$\begin{cases} q(0) = q_0 \\ I(0) = I_0 \end{cases}$$

* <u>Differentiate both sides of the equation above and you'll get an ODE for $I(t)$.</u>

2^{nd} order constant coefficients :

$$y'' + by' + cy = 0$$

Convert to a 1^{st} order system :

$$y = u \Rightarrow y' = u' = v$$

$$y' = v \Rightarrow y'' = v' = -by' - cy = -bv - cu$$

$$\begin{cases} u' = v \\ v' = -cu - bv \end{cases} \Rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}\begin{pmatrix} \overset{\nearrow y}{u} \\ \underset{\searrow y'}{v} \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$: Solve the system with eigval and eigvec method

$$\det \begin{pmatrix} -r & 1 \\ -c & -b-r \end{pmatrix} = r(r+b) + c = 0$$

$$\Rightarrow \boxed{r^2 + br + c = 0}$$ → Characteristic equation for a 2nd order of the form

Compare this to $\boxed{y'' + by' + cy = 0}$

Example : $y'' + 3y' + 2y = 0$

$$\begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$\xrightarrow{\text{characteristic equation}}$ $r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1, \; r_2 = -2$

$$\Rightarrow r_1 = -1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$r_2 = -2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{-2t} \\ -c_1 e^{-t} - 2c_2 e^{-2t} \end{pmatrix}$$

Only take the 1st component of the solution vector as the solution of the 2nd order ODE.

The 2nd component is just the derivative.

$\xrightarrow{\text{General solution}}$ $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

In general:

$$y'' + by' + cy = 0$$

$$r^2 + br + c = 0$$

solve: eigval $r_{1,2} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2-4c}$

if r_1 and r_2 are real and distinct $(b^2-4c>0)$

$$\Rightarrow \boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

if $r_{1,2} = \alpha \pm i\beta \quad (b^2-4c<0)$

Do the computation with the system form and take the 1st component

$$\Rightarrow \boxed{y(t) = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t}$$

if $r^2 + br + c = 0$ has a double root r

repeated eigval

Defective matrix

$$\Rightarrow \boxed{y(t) = c_1 e^{rt} + c_2 t e^{rt}}$$

→ Do the computation with the system representation and take the 1st component.

Ex.

$$\begin{cases} y'' + 3y' = 0 \\ y(0) = -2 \\ y'(0) = 3 \end{cases}$$

$y'' + 3y' = 0 \longrightarrow r^2 + 3r = 0 \Rightarrow r = 0, -3$

$\Rightarrow \boxed{y(t) = c_1 + c_2 e^{-3t}} \Rightarrow y(t) = -1 - e^{-3t}$

$y(0) = c_1 + c_2 = -2 \Rightarrow \boxed{c_1 = -2 + 1 = -1}$

$y'(t) = -3c_2 e^{-3t} \longrightarrow y'(0) = -3c_2 = 3 \Rightarrow \boxed{c_2 = -1}$