HW 3 — Oct 26, Lecture 22

Q2: initial vector at $x(0)$

$\hookrightarrow$ means: initial tangent vector: $x'(0)$

Last day:

2^{nd} order linear hom with constant coeff:

$$y'' + by' + cy = 0$$

characteristic equation $\quad r^2 + br + c = 0$

by converting to: $\begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$

We applied eigval & eigvec method to solve the system and we picked the 1^{st} entry in the vector of system solution.

$r^2 + br + c = 0$

- $\to$ two distinct real $r_{1,2} \rightsquigarrow y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- $\to$ complex conjugate $r = \alpha \pm i\beta \rightsquigarrow y(t) = c_1 e^{\alpha t} \sin\beta t + c_2 e^{\alpha t} \cos\beta t$
- $\to$ double root r $\rightsquigarrow y(t) = c_1 e^{rt} + c_2 t e^{rt}$

Ex 1. Solve $\quad y'' - 2y' + 10y = 0, \quad y(0) = -3, \quad y'(0) = 1$

$$r^2 - 2r + 10 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \overset{\alpha}{1} \pm \overset{\beta}{3} i$$

$$y(t) = c_1 e^t \sin 3t + c_2 e^t \cos 3t$$

$$y'(t) = c_1 e^t \sin 3t + c_1 e^t \, 3\cos 3t + c_2 e^t \cos 3t + c_2 e^t \, {-3}\sin 3t$$

$$y(0) = c_2 = -3$$

$$y'(0) = 3c_1 + c_2 = 1 \xrightarrow{c_2=-3} 3c_1 = 1+3 \Rightarrow c_1 = \frac{4}{3}$$

Physical Interpretation of

$$y'' + by' + cy = 0 \qquad b, c > 0$$

$$r^2 + br + c = 0$$

$$r_{1,2} = \frac{-b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$$

(1) $b = 0$ (undamped case (No friction))

$$y'' + cy = 0$$

$$r^2 + c = 0 \overset{c>0}{\Rightarrow} r = \pm i\sqrt{c}$$

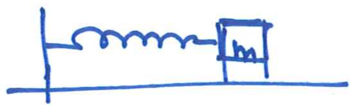
$$y(t) = c_1 \cos(\overset{(\omega t)}{\sqrt{c}\,t}) + c_2 \sin(\overset{(\omega t)}{\sqrt{c}\,t}) \quad (\mathrm{I})$$

pure oscillation

$\sqrt{c}$ is called the natural frequency (or resonant frequency).

notation: ω unit: Rad/sec

Mass-spring system :



$$x'' + \frac{k}{m} x = 0$$

$$\omega = \sqrt{c} = \sqrt{\frac{k}{m}} \rightarrow \text{natural freq}$$

RLC circuit

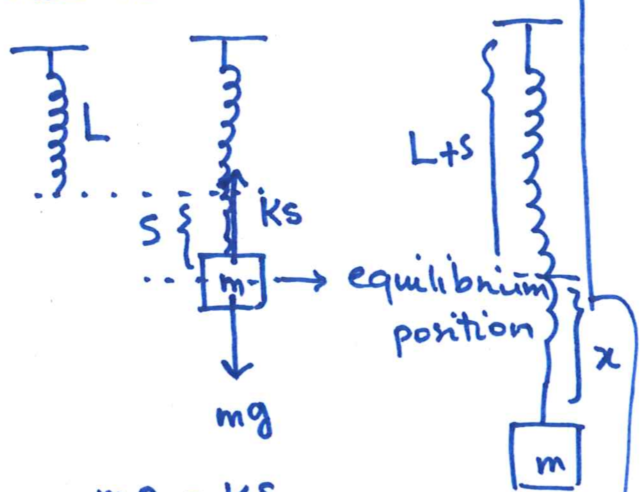
$$LQ'' + R\cancel{Q'} + \frac{1}{C} Q = 0$$

0

undamped → No resistor $\Rightarrow LQ'' + \frac{1}{C} Q = 0$

$$\Rightarrow \omega = \sqrt{\frac{1}{LC}}$$

Side note :



$mg = KS$

x : displacement from the equilibrium

It is convienient to write $y(t)$

as $y(t) = R\cos(\omega t - \theta)$ (II)

Use the trig identity :

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Equate (I) and (II) to find

R and θ in terms of c_1 and c_2

then :

$$R = \sqrt{c_1^2 + c_2^2}$$

and $\theta = \arctan\left(\frac{c_1}{c_2}\right)$

$$y(t) = R\cos(\omega t - \theta)$$

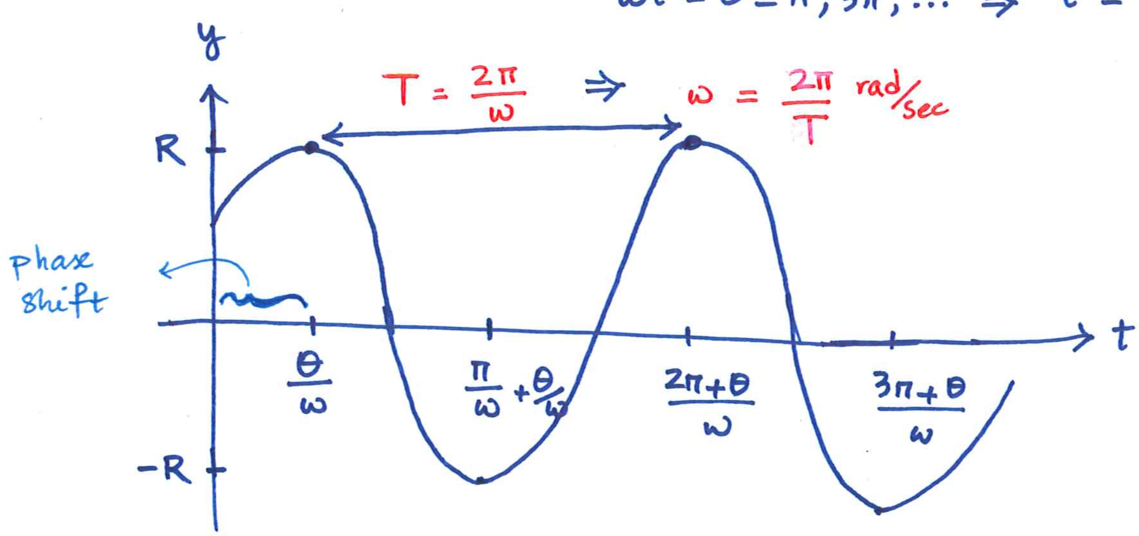
$$= \sqrt{c_1^2 + c_2^2}\,\cos\left(\omega t - \arctan\left(\frac{c_1}{c_2}\right)\right)$$

Plot $y(t) = R\cos(\omega t - \theta)$

* $\cos(\omega t - \theta) = 1$ when $\omega t - \theta = 0, 2\pi, \dots \Rightarrow t = \frac{\theta}{\omega}, \frac{2\pi+\theta}{\omega}, \frac{4\pi+\theta}{\omega}, \dots$
* $\cos(\omega t - \theta) = 0$ when $\omega t - \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

  $\Rightarrow t = \frac{\frac{\pi}{2}+\theta}{\omega}, \frac{3\pi/2+\theta}{\omega}, \dots$
* $\cos(\omega t - \theta) = -1$ when

  $\omega t - \theta = \pi, 3\pi, \dots \Rightarrow t = \frac{\pi+\theta}{\omega}, \frac{3\pi+\theta}{\omega}, \dots$

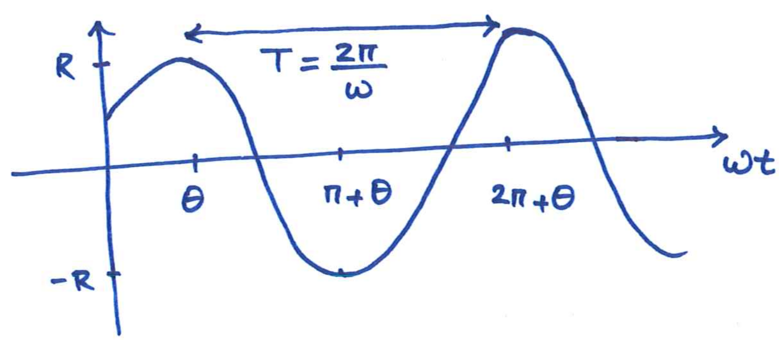


T : period $= \frac{2\pi}{\omega}$

ω : natural freq $= \frac{2\pi}{T}$

R = Maximum displacement = Amplitude

or rescale the t-axis to ωt



* R is fixed in an undamped system

  $\Rightarrow$ Energy given to the system initially by $x(0)$ and $v(0)$ NEVER gets dissipated.

* θ is called phase (or angular phase.)

(2) $b \neq 0$, $b > 0$ $\quad y'' + by' + cy = 0 \quad r_{1,2} = -b/2 \pm \frac{\sqrt{b^2 - 4c}}{2}$

(I) $b^2 - 4c < 0 \rightarrow r_{1,2} = -\overset{>0}{b}/2 \pm i \underbrace{\frac{\sqrt{4c - b^2}}{2}}_{\beta}$

$$y(t) = c_1 e^{\overset{<0}{(-b/2)} t} \cos \beta t + c_2 e^{-b/2 t} \sin \beta t$$

Oscillation occurs

all solution $\longrightarrow 0$ as $t \rightarrow \infty$

This case is called underdamping (weak damping)

(II) $b^2 - 4c > 0$

$b, c > 0 \quad r_{1,2} = -b/2 \pm \frac{\sqrt{b^2 - 4c}}{2} < 0$

$$r_{1,2} < 0 \begin{cases} b^2 - 4c < b^2 \\ \frac{\sqrt{b^2 - 4c}}{2} < \frac{b}{2} \end{cases}$$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \xrightarrow{\text{as } t \to \infty} 0$$

NO oscillation

This case is called overdamping (Strong damping)

(III) $b^2 - 4c = 0 \Rightarrow r = -b/2$

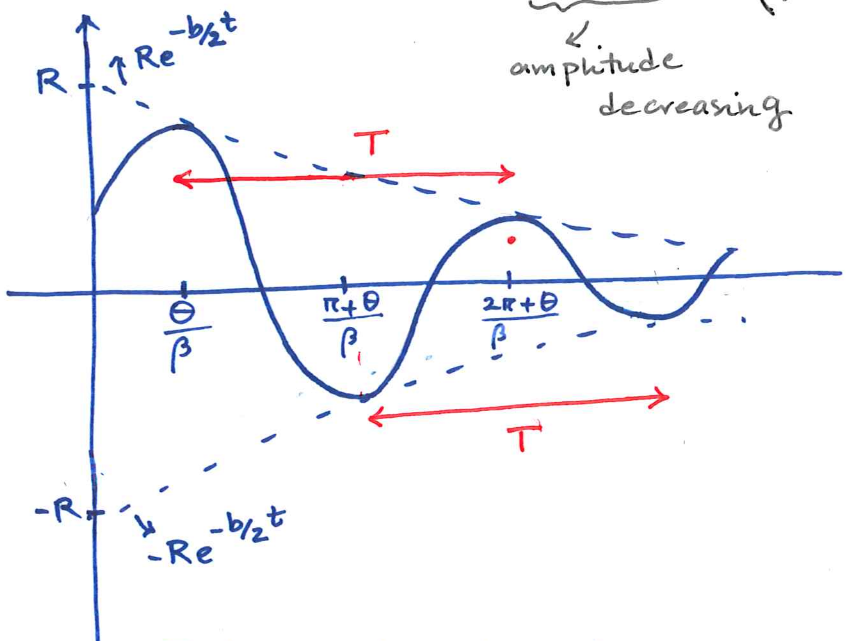
$$\Rightarrow y(t) = c_1 e^{-b/2\, t} + c_2 t e^{-b/2\, t}$$

This is called the critically damped case: The weakest possible damping that prevents oscillation below which oscillation occurs.

Analyze case (2.I)

$$b^2 - 4c < 0 \Rightarrow y(t) = c_1 e^{-b/2\, t} \cos \beta t + c_2 e^{-b/2\, t} \sin \beta t$$

$$\Rightarrow y(t) = \underbrace{R e^{-b/2\, t}}_{\text{amplitude decreasing}} \cos(\beta t - \theta)$$



$$\beta = \frac{\sqrt{4c - b^2}}{2} = \sqrt{c - \frac{b^2}{4}}$$

fix this in your class note: frequency in the damped case remains fixed over time.

In fact, Energy dissipation only affects the amplitude.

But as damping increases, frequency decreases.

Undamped : $\omega = \sqrt{c}$

damped : $\beta = \omega_d = \sqrt{c - b^2/4} < \omega$

$$y'' + by' + cy = 0$$

$\rightsquigarrow$ increase b $\rightsquigarrow \omega_d$ gets smaller.

$\rightsquigarrow$ small damping $\sqrt{c} \approx \sqrt{c - b^2/4} \Rightarrow \omega \approx \omega_d$

- Questions :

Mass - spring

$T = \frac{2\pi}{\omega}$, $\omega = \sqrt{\frac{K}{m}}$

(1) mass increases → $T = ?$, $\omega = ?$

(2) spring gets stiffer → $T = ?$, $\omega = ?$

m increases → ω decreases → period increases

→ slower oscillations .

spring stiffer → k increasing → ω increase

→ T decrease

→ faster oscillations