Oct 29
Lecture 23

Last Class : (some corrections)

$$y'' + by' + cy = 0 \qquad b, c > 0$$

$$r^2 + br + c = 0$$

$$r_{1,2} = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

(1) $\underline{b = 0}$ $\Rightarrow$ $r_{1,2} = \pm i\sqrt{c} = \pm i\omega$ ← natural frequency

undamped case

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

$$= R \cos(\omega t - \theta)$$

R → amplitude, θ → phase

$$R = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \theta = \arctan \frac{c_1}{c_2}$$

$y(t) = R \cos(\omega t - \theta)$ $\xrightarrow{\text{How to plot:}}$ $*$ $\cos(\omega t - \theta) = 1$

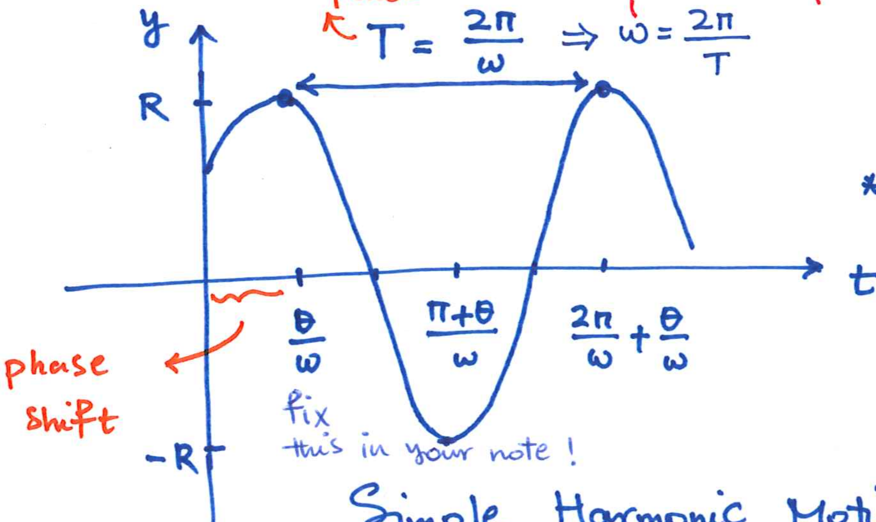
period ← $T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$ ← natural freq.

$\omega t - \theta = 0 \Rightarrow t = \frac{\theta}{\omega}$

$= 2\pi \Rightarrow t = \frac{2\pi + \theta}{\omega}$

$= 4\pi$

$\vdots$



$*$ $\cos(\omega t - \theta) = -1$

$\omega t - \theta = \pi$

$t = \frac{\pi + \theta}{\omega}$

Simple Harmonic Motion

(2) $b \neq 0$

(I) $b^2 - 4c > 0$ $\Rightarrow$ $r_{1,2} = -b/2 \pm \overbrace{\frac{\sqrt{b^2-4c}}{2}}^{\beta} < 0$

$b, c > 0$

Over-damped case

$y(t) = c_1 e^{-b/2 t + \beta t} + c_2 e^{-b/2 - \beta t}$

$= c_1 e^{r_1 t} + c_2 e^{r_2 t} \xrightarrow{t \to \infty} 0$

(II) $b^2 - 4c = 0$ $\Rightarrow$ $r_{1,2} = -b/2 = r < 0$

Critically damped

$\Rightarrow y(t) = c_1 e^{rt} + c_2 t e^{rt} \xrightarrow{t \to \infty} 0$

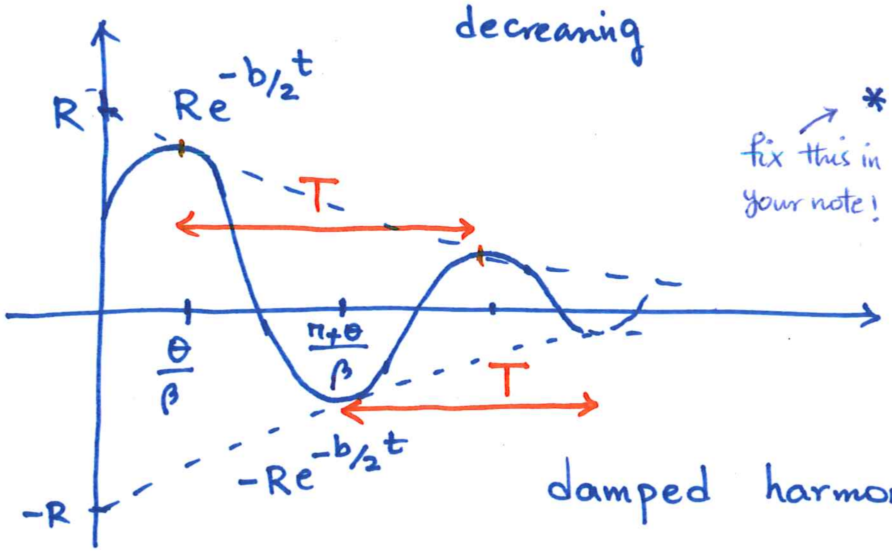
(III) $b^2 - 4c < 0$ $\Rightarrow$ $r_{1,2} = -b/2 \pm i \overbrace{\frac{\sqrt{4c - b^2}}{2}}^{\beta}$

Underdamping

$\Rightarrow y(t) = c_1 e^{-b/2 t} \cos \beta t + c_2 e^{-b/2} \sin \beta t$

$= R e^{-b/2 t} \cos(\beta t - \theta)$

amplitude decreasing



fix this in your note!

* T is fixed during the motion of the object & frequency $\beta = \sqrt{c - \frac{b^2}{4}}$ is fixed over time.

damped harmonic motion

Note: If we re-run the experiment with increasing damping force $\rightsquigarrow$ b increases

then frequency decreases.

Compare undamped frequency $\omega = \sqrt{c}$

and damped " $\beta = \omega_d = \sqrt{c - b^2/4} < \sqrt{c} = \omega$

as b increases, ω_d will decrease.

- Assuming that damping force is small (b is small)

then $\sqrt{c} \approx \sqrt{c - b^2/4}$

$$\Rightarrow \underbrace{\omega}_{\text{natural frequency (undamped)}} \approx \omega_d = \beta$$

$\underbrace{\text{Nonhomogeneous}}_{\text{forced}}$ 2^{nd} order linear ODE

$$y'' + by' + cy = g(t)$$

general solution: $y(t) = y_H(t) + y_p(t)$

where y_H is a general solution of the hom. eqt:

y_H has constants c_1 & c_2

$$y'' + by' + cy = 0$$

and y_p is a particular solution of

$$y'' + by' + cy = g(t)$$

Finding $y_p(t)$:

two methods: $\rightsquigarrow$ Similar to systems

1. $\boxed{\text{Variation of parameters}}$: Find fundamental matrix corresponding to hom. part $\vec{x}' = A\vec{x}$

Rewrite as a system:

$$\underbrace{\begin{pmatrix} y' \\ y'' \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y \\ y' \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} 0 \\ g(t) \end{pmatrix}}_{G}$$

$$X_p(t) = \mathbb{X}(t) \int \mathbb{X}^{-1}(t)\, G(t)\, dt = \begin{pmatrix} \boxed{\ \cdots\ } \\ \cdots \end{pmatrix} \to y_p(t)$$

* Cons: lots of matrix algebra and also integration.
* pros: works for any 2^{nd} order forced ODE regardless of the type of $g(t)$.

2. $\boxed{\text{Undetermined coefficients}}$:

We guess a particular solution with unknow coefficients

↳ plug your guess into the equation and find the coefficients.

* Cons: does NOT work for all $g(t)$.

works if only $g(t)$ is constant, polynomial, exp, sin and Cos

* pros: Usually easier computations comparing to method 1.
Shorter

undetermined coefficients does NOT work here!

Ex 1: $y'' + 4y = 8\tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 8\tan t \end{pmatrix}}_{G(t)}$$

$$\det\begin{pmatrix} -r & 1 \\ -4 & -r \end{pmatrix} = 0 \Rightarrow r^2+4=0 \Rightarrow r = \pm 2i \Rightarrow \text{Find } \mathbb{X}(t) = ?$$

$r_1 = 2i$
$$\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 2iv_1 = v_2 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$X(t) = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}(\cos 2t + i\sin 2t)$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ -2\sin 2t + i\, 2\cos 2t \end{pmatrix}$$

$$= \underbrace{U(t)} + i\,\underbrace{V(t)}$$

columns of $\mathbb{X}(t)$

$$\mathbb{X}(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix}$$

$$\mathbb{X}^{-1}(t) = \frac{1}{\underbrace{2\cos^2 2t + 2\sin^2 t}_{2}}\begin{pmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix} = \begin{pmatrix} \cos 2t & -\frac{1}{2}\sin 2t \\ \sin 2t & \frac{1}{2}\cos 2t \end{pmatrix}$$

$$\int X^{-1} G \, dt = 4 \int \begin{pmatrix} -\sin 2t \cdot \tan t \\ \cos 2t \cdot \tan t \end{pmatrix}$$

$* \sin 2t = 2 \sin t \cos t$

$* \cos 2t = 2\cos^2 t - 1$

$= 1 - 2\sin^2 t$

use trig identities to simplify the integrand

$$= 4 \int \begin{pmatrix} \cos 2t - 1 \\ \sin 2t - \tan t \end{pmatrix}$$

$$= 4 \begin{pmatrix} \frac{1}{2}\sin 2t - t \\ -\frac{1}{2}\cos 2t + \ln(\cos t) \end{pmatrix}$$

$$X_p = X \int X^{-1} G = 4 \begin{pmatrix} -t\cos 2t + \sin 2t \ln(\cos t) \\ \text{------} \end{pmatrix}$$

So $y_p(t) = \underbrace{4\sin 2t \ln(\cos t) - 4t\cos 2t}$

y_H solves $y'' + 4y = 0$

$\Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\Rightarrow y_H(t) = c_1 \cos 2t + c_2 \sin 2t$

General Solution: $y(t) = c_1 \cos 2t + c_2 \sin 2t +$

Ex 2 : $y'' + 4y = t^3$ a polynomial, we can apply the 2^{nd} method.

Our guess : $y_p(t) = At^3 + Bt^2 + Ct + D$

$y_p'(t) = 3At^2 + 2Bt + C$

$y_p''(t) = 6At + 2B$

plug into eqt.

$y_p'' + 4y_p = 6At + 2B + \underline{4At^3} + \underline{4Bt^2} + 4Ct + 4D$

$= t^3$

$4A = 1 \Rightarrow A = \frac{1}{4}$

$4B = 0 \Rightarrow B = 0$

$6A + 4C = 0 \Rightarrow C = -\frac{6A}{4} = -\frac{6}{16} = -\frac{3}{8}$

$2B + 4D = 0 \xRightarrow{B=0} D = 0$

$\Longrightarrow y_p(t) = \frac{1}{4}t^3 - \frac{3}{8}t$

General solution : $y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4}t^3 - \frac{3}{8}t$