Nov 2

Lecture [redacted]

forced 2^{nd} order ODE $\quad y'' + by' + cy = g(t)$

$$y(t) = y_H(t) + y_p(t)$$

Undetermined coefficients :

we guess y_p $\longrightarrow$ plug into the eqt

$\longrightarrow$ Find coefficient

$g(t)$ given	$y_p(t)$ unknown
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ (given)	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$ (unknown)
$e^{\alpha t}$ (α given)	$A e^{\alpha t}$ (A unknown)
$\sin \beta t$ or $\cos \beta t$ (β given)	$A \sin \beta t + B \cos \beta t$ (A, B unknown)
$P_n(t) \cdot e^{\alpha t} \cdot \sin \beta t$ or $P_n(t) \cdot e^{\alpha t} \cdot \cos \beta t$	$(A_n t^n + \dots + A_1 t + A_0) e^{\alpha t} \sin \beta t + (B_n t^n + \cdots + B_1 t + B_0) e^{\alpha t} \cos \beta t$

multiply y_p by t when :

$g(t)$ → If $g(t)$ is itself a solution of the homog. eqt.	$y_p(t)$
$P_n(t)$ and 0 is a root of the characteristic eqt → This is $r^2+br+c=0$	$t\left(A_n t^n + \cdots + A_1 t + A_0\right)$
$e^{\alpha t}$ and α is a root of the char eqt	$Ate^{\alpha t}$
$\cos\beta t$ or $\sin\beta t$ and $\pm i\beta$ is a root of char eqt	$At\cos\beta t + Bt\sin\beta t$
$e^{\alpha t}\cos\beta t$ or $e^{\alpha t}\sin\beta t$ and $\alpha \pm i\beta$ a root of char eqt	$Ate^{\alpha t}\cos\beta t + Bte^{\alpha t}\sin\beta t$

<u>Ex1</u>. $y'' + 2y' + y = e^{-t}$

initial guess: Ae^{-t}

$r^2 + 2r + 1 = 0$ $\xrightarrow[\text{2nd guess}]{\text{modify}}$ $y_p = Ate^{-t}$

$\Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$

$\Rightarrow y_H(t) = \boxed{c_1 e^{-t}} + \boxed{c_2 te^{-t}}$

initial guess 2nd guess

still does not work

$\xrightarrow[\text{3rd guess}]{\text{modify}}$ $y_p(t) = At^2 e^{-t}$

You do the algebra! $A = \frac{1}{2}$

Note: If $g(t)$ is sum of two functions:

$$y'' + by' + cy = g_1(t) + g_2(t)$$

then

$$y'' + by' + cy = g_1(t) \longrightarrow \text{find } y_{p_1} \text{ for } g_1$$

$$y'' + by' + cy = g_2(t) \longrightarrow \text{find } y_{p_2} \text{ for } g_2$$

and the particular solution is

$$Y_p(t) = y_{p_1}(t) + y_{p_2}(t)$$

This method works if $g(t) = g_1(t) + g_2(t) + \cdots + g_n(t)$

Exercise: $\underline{y'' - 3y' - 4y} = 3e^{2t} + 2\sin t - 8e^t \cos 2t$

$\hookrightarrow r^2 - 3r - 4 = 0 \Rightarrow r = -1, 4$

- $y'' - 3y' - 4y = 3e^{2t}$

  $$\Rightarrow y_{p_1}(t) = Ae^{2t} \xrightarrow{\text{Find } A}$$

- $y'' - 3y' - 4y = 2\sin t$

  $$\Rightarrow y_{p_2}(t) = B\sin t + D\cos t \xrightarrow{\text{Find } B \text{ and } D}$$

- $y'' - 3y' - 4y = -8e^t \cos 2t$

  $$\Rightarrow y_{p_3}(t) = Ee^t\cos 2t + Fe^t \sin 2t \xrightarrow{\text{Find } E \text{ and } F}$$

$\Rightarrow Y_p(t) = y_{p_1} + y_{p_2} + y_{p_3}$ and $y(t) = c_1 e^{-t} + c_2 e^{4t} + Y_p(t)$

Forced Vibrations :

Consider an undamped mass-spring system :

$$m x'' + k x = g(t) \qquad m, k > 0$$

x_H : $\quad m x'' + k x = 0$

$$m r^2 + k = 0 \Rightarrow r = \pm \sqrt{\frac{k}{m}}\, i = \pm \omega_0 i$$

natural frequency

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

forcing frequency

Consider $g(t) = F_0 \cos(\omega t)$

constant

$$m x'' + k x = F_0 \cos(\omega t)$$

To find $x_p(t)$, we need to consider two cases :

(1) $\omega \neq \omega_0$ (ω is not a root of char. eqt.)

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$x_p'(t) = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

$$x_p''(t) = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)$$

$$m x_p'' + k x_p = F_0 \cos(\omega t)$$

$$-Am\omega^2 \cos(\omega t) - Bm\omega^2 \sin(\omega t) + Ak\cos(\omega t) + Bk\sin(\omega t)$$

$$= F_0 \cos(\omega t)$$

$$B\left(k - m\omega^2\right) = 0 \Rightarrow B = 0$$

$$A\left(k - m\omega^2\right) = F_0 \Rightarrow A = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

Remember $\omega_0 = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$\Rightarrow \quad x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Assume the motion starts at equilibrium:

$$x(0) = 0 \ , \quad x'(0) = 0$$

$$\xrightarrow{x(0)=0} 0 = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} \Rightarrow c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}$$

$$\xrightarrow{x'(0)=0} 0 = c_2 \omega_0 \Rightarrow c_2 = 0$$

$$\Rightarrow \quad x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t) \right)$$

Use trig identities:

$$\begin{cases} * \ \cos(A+B) = \cos A \cos B - \sin A \sin B \\ * \ \cos(A-B) = \cos A \cos B + \sin A \sin B \end{cases}$$

$$\Rightarrow \cos(A-B) - \cos(A+B) = 2 \sin A \sin B$$

if: $A - B = \omega t$, $A + B = \omega_0 t$ $\Rightarrow$ $A = \dfrac{\omega + \omega_0}{2} t$, $B = \dfrac{\omega_0 - \omega}{2} t$

$$\Rightarrow \boxed{x(t) = \boxed{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right)} \sin\left(\frac{\omega_0 + \omega}{2} t\right)}$$

large frequency

amplitude → represents a periodic change in amplitude

if ω and ω_0 are close to each other:

$\omega - \omega_0$ is small

$\omega + \omega_0$ is big

This phenomenon is called $\boxed{\text{beats}}$.

<u>Application</u>: Tuning a musical instrument, radar Technology

↓

The tuner and the piano string sound simultaneously

↓

beats will form due to difference in frequencies

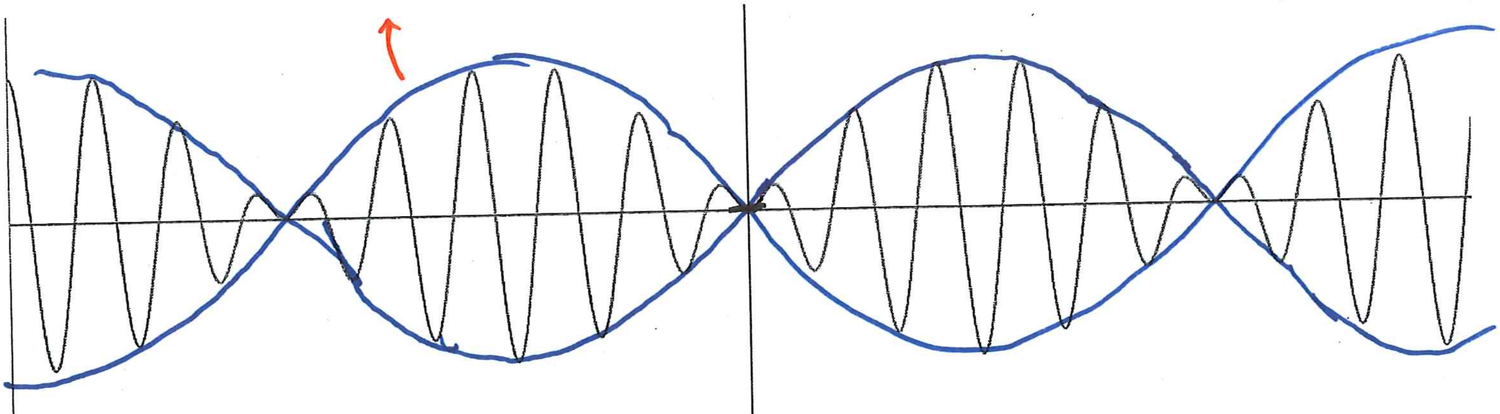
↓

do the tuning

↓

The slower the beats, the more nearly the string is tuned.

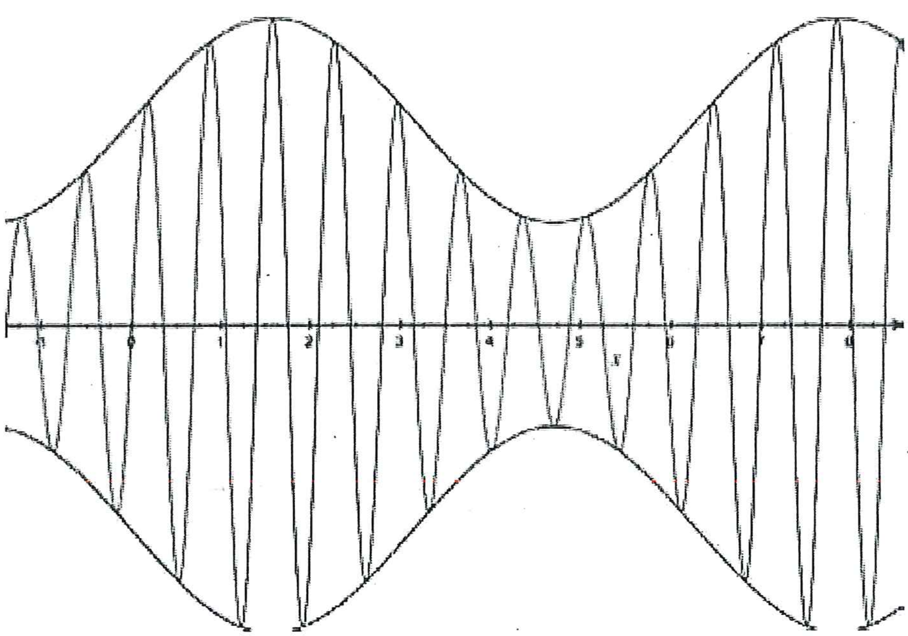
the periodic modulation in amplitude.



beats with zero initial condition

$x(0)=0 \quad , \quad x'(0)=0$

beats with non-zero initial condition.



(2) if $\omega = \omega_0 \to \omega$ is a root of the char. eqt :

$$x_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$$

You do the algebra of finding x' and x''

$$= F_0 \cos(\omega t)$$

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}$$

$$x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F}{2m\omega_0} t\sin(\omega t)$$

if $x(0)=0$, $x'(0)=0$

$$\Rightarrow \quad c_1 = c_2 = 0$$

$$\Rightarrow \quad x(t) = \frac{F}{2m\omega_0} t\sin(\omega t)$$

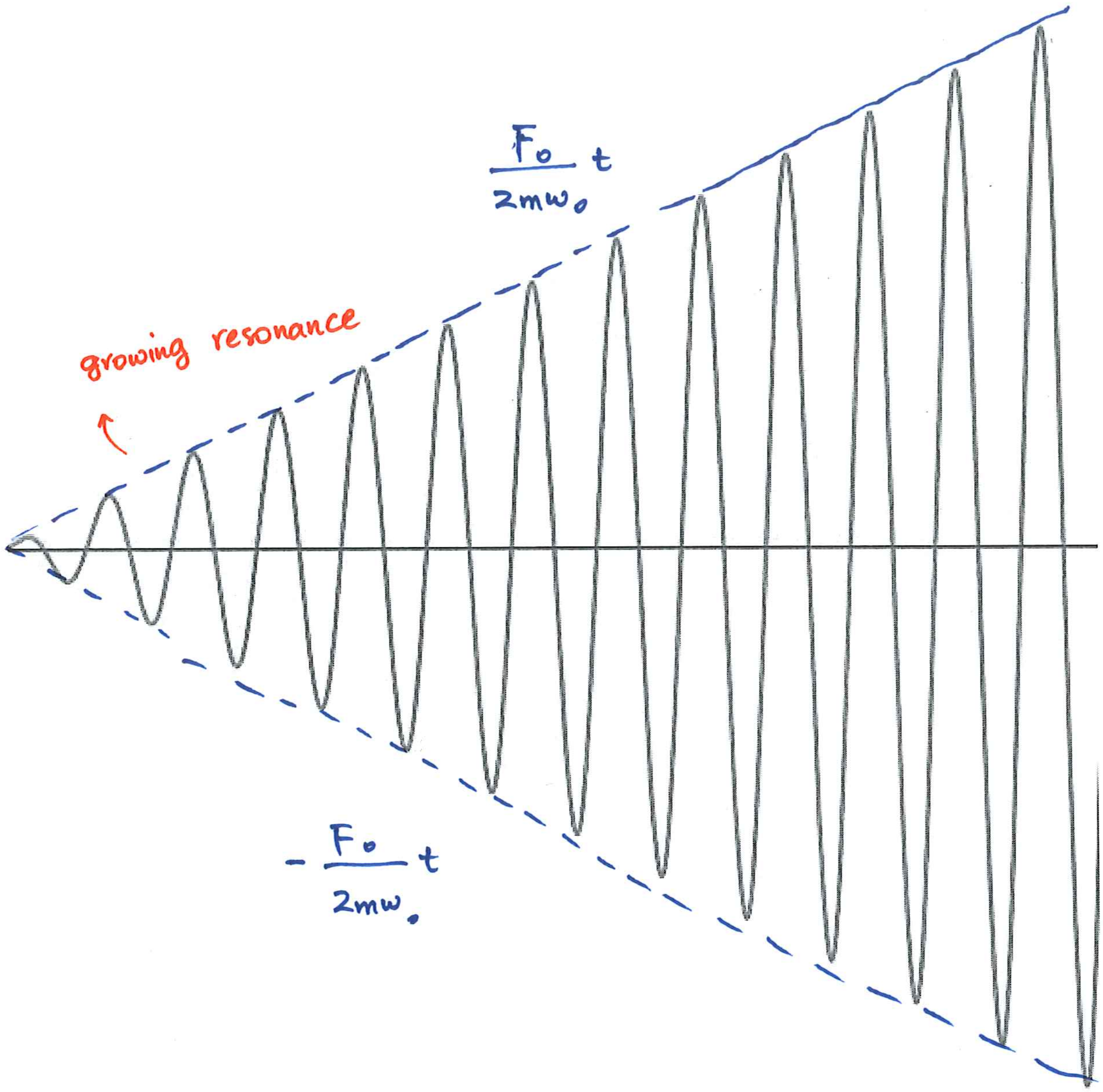
Resonance phenomenon $\leftarrow$ amplitude : is linearly growing with t

when $t \to \infty$: $x(t) \to \infty$

Resonance is unbounded.

Significantly important in engineering.

* We must avoid resonance at any cost!



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