Nov 5

Lecture 26

Last Class :

Forced Vibrations (undamped)

$$m x'' + k x = F_0 \cos(\omega t)$$

$$x(t) = x_H(t) + x_p(t)$$

natural freq.

$$m r^2 + k = 0 \Rightarrow r_{1,2} = \pm\sqrt{\frac{k}{m}}\, i = \pm \omega_0 i$$

$$x_H(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Undeter. Coeff:

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t) \quad \text{if } \boxed{\omega \neq \omega_0}$$

$$\Rightarrow B = 0, \quad A = \frac{+F_0}{m(\omega_0^2 - \omega^2)}$$

if $x(0) = 0$, $x'(0) = 0$

So $$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

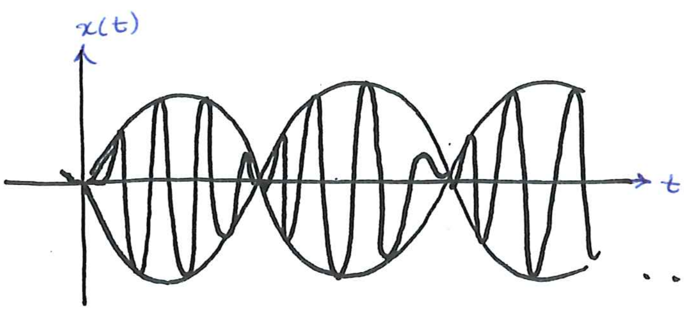
trig identity

$$= \boxed{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\Big(\underbrace{\frac{\omega_0 - \omega}{2}}_{\text{lower frequency}} t\Big)} \sin\Big(\underbrace{\frac{\omega_0 + \omega}{2}}_{\text{higher frequency}} t\Big)$$

if ω and ω_0 are close then

represents a periodic amplitude

$\Rightarrow$ Beats :



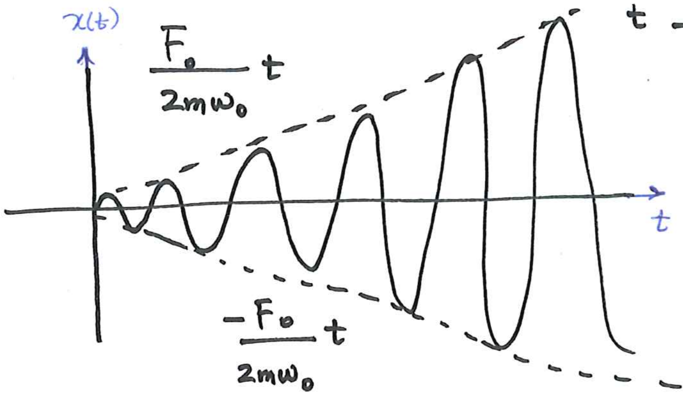
...

if $\omega = \omega_0 \Rightarrow x_p(t) = At\cos(\omega t) + Bt\sin(\omega t)$

$$A = 0, \quad B = \frac{F_0}{2m\omega_0}$$

$$\left.\begin{array}{l} x(0)=0 \\ x'(0)=0 \end{array}\right. \xrightarrow{c_1 = c_2 = 0} \quad x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega t)$$

linearly growing amplitude $\Rightarrow$ Resonance



$t \to \infty$, $x(t) \to \infty$

Test Yourself:

Consider a mass-spring system described by the equation

$$x'' + 9x = F\cos(\omega t)$$

$m = 1$
$k = 9$ $\Rightarrow \omega_0 = \sqrt{\frac{k}{m}} = 3$

Match the given values for ω to the graph of the motion of the object.

For example:

A. $\omega = 3$ → Resonance (4)

3 ← B. $\omega = 3.1$ → $\sin\left(\frac{\omega - \omega_0}{2} t\right)$

2 ← C. $\omega = 3.3$

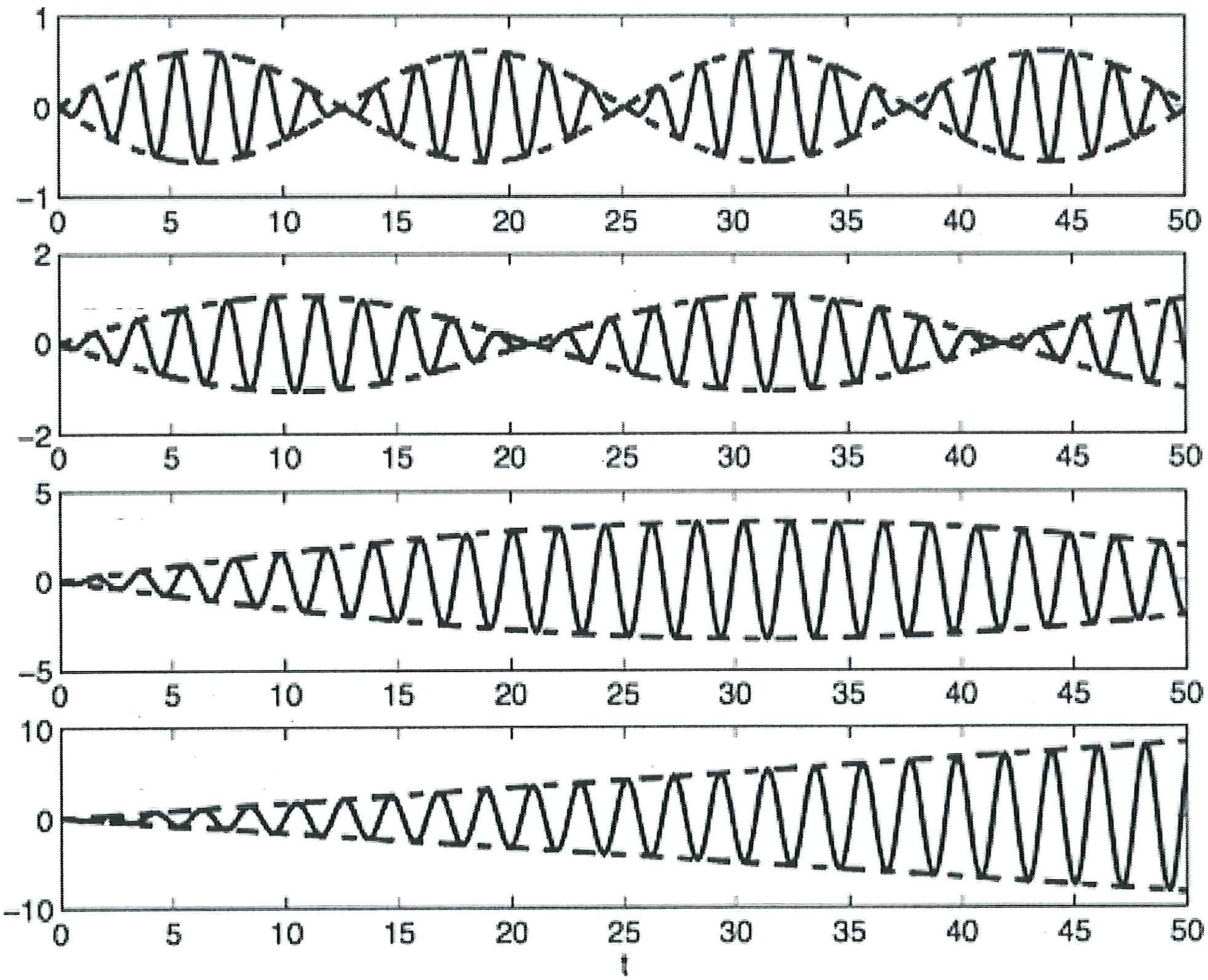
1 ← D. $\omega = 3.5$

as ω increase $\Rightarrow$ $\omega - \omega_0$ increases $\Rightarrow$ period decreases

ω INC: $\sin(2t)$, $\sin(4t)$, $\sin(10t)$ — T DEC

Beats and Resonance

1.
2.
3.
4.



Damped forced vibrations:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

$m, c, k > 0$

$\omega_0 = \sqrt{\frac{k}{m}}$

$$r_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

$r_{1,2}$ always negative

$$x_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

as $t \to \infty$: $x_H \longrightarrow 0$

transient solution.

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

* Think: why don't we multiply by t?

x_p' , $x_p'' \longrightarrow$ plug into the equation

$\longrightarrow$ Find $\underline{A, B}$

ugly constant in terms of m, ω, ω_0, F_0 and c

Steady-state sol'n or forcing response

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$= R \cos(\omega t - \theta)$$

as $t \to \infty$, x_p keeps oscillating

amplitude

$$R = \frac{F_0}{\sqrt{m^2(\omega^2 - \omega_0^2)^2 + c^2\omega^2}}$$

Full general solution:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + A \cos(\omega t) + B \sin(\omega t)$$

$$= c_1 e^{r_1 t} + c_2 e^{r_2 t} + R \cos(\omega t - \theta)$$

$t \to \infty$: $x_H \to 0$ so $x(t)$ behaves similar to the steady-state solution; it keeps oscillating.

<u>Ex.</u> $x'' + 2x' + 10x = \cos(2t)$

$x(0) = -\frac{1}{2}$, $x'(0) = 4$

$m = 1$
$c = 2 \Rightarrow \omega_0 = \sqrt{10}$
$k = 10$

$r^2 + 2r + 10 = 0 \Rightarrow r = -1 \pm 3i \Rightarrow$

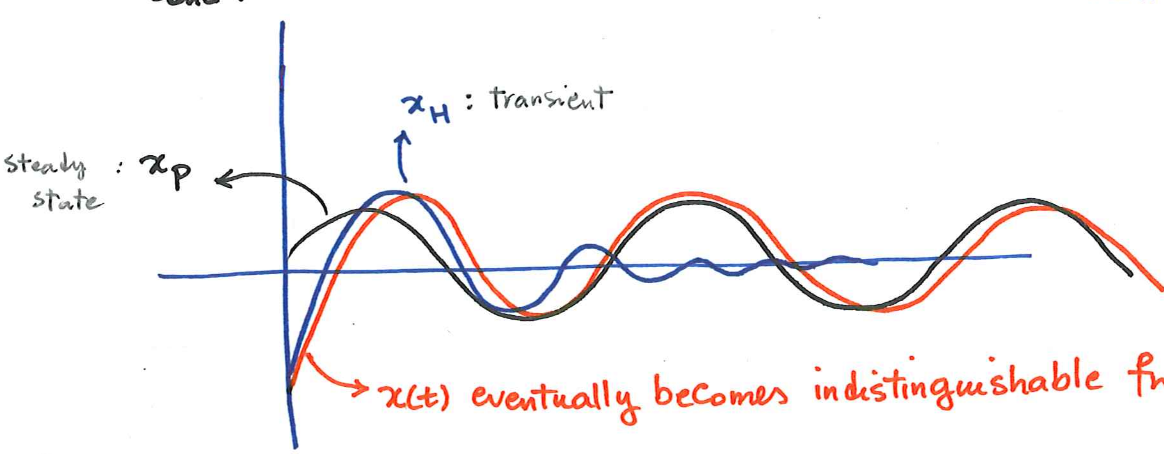
$\Rightarrow \boxed{x_H(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t} \rightarrow$ transient sol'n

steady-state response

$\boxed{x_p(t) = A \cos(2t) + B \sin(2t)} \Rightarrow A = 3/26$, $B = 1/13$

$\Rightarrow \boxed{x(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t + 3/26 \cos(2t) + 1/13 \sin(3t)}$ → Full solution

$\xrightarrow[\text{Cond.}]{\text{Initial}}$ $c_1 = -8/13$, $c_2 = 1$



$x(t)$ eventually becomes indistinguishable from the force response.

R as a function of ω, study R for different c values.

$$R(\omega) = \frac{F_0}{\sqrt{m^2(\omega^2-\omega_0^2)^2 + c^2\omega^2}}$$

when ω small i.e. $\omega \to 0$; $R \to \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$

ω large $\omega \to \infty$; $R \to 0$

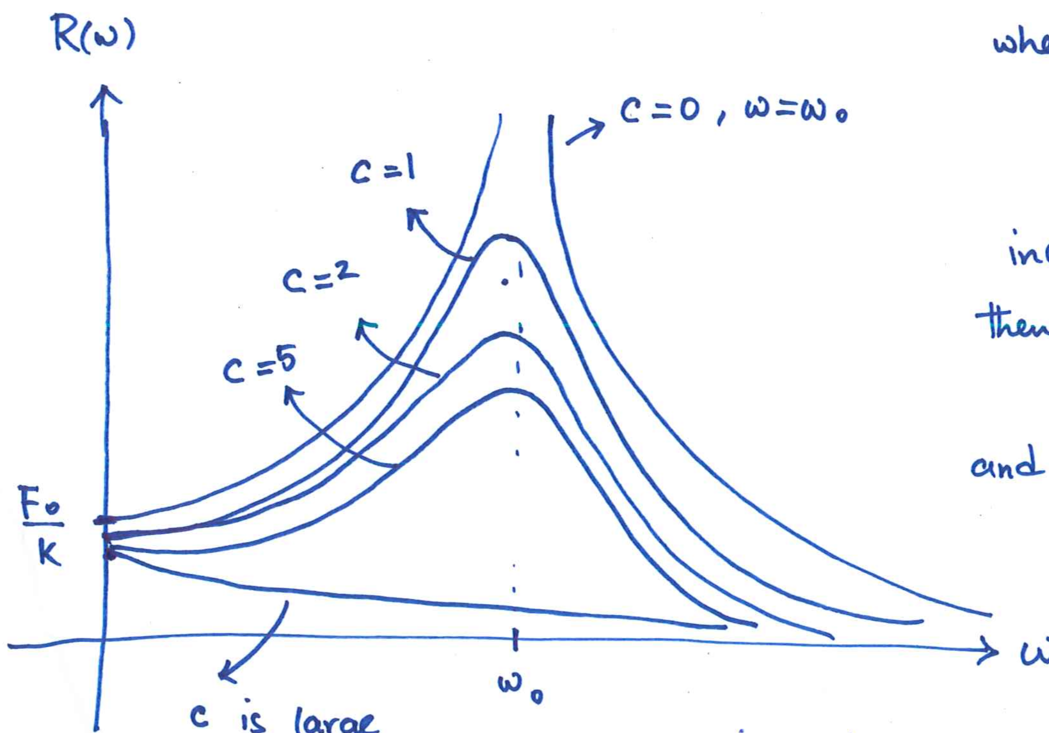
when $c = 0$ (no damping)

$\omega = \omega_0$ then $R \to \infty$

($\omega = \omega_0$ vertical asymp. for R)

ω_{max} occurs when $R'(\omega) = 0$

$$\Rightarrow \omega_{max}^2 = \omega_0^2\left(1 - \frac{c^2}{2mk}\right) < \omega_0^2$$



when c small

$\omega_{max} \approx \omega_0$

increase c

then ω_{max} becomes imaginary

and max value occurs

when $\omega = 0$

$\to$ $R(\omega)$ strictly decreasing

* c increasing $\rightsquigarrow$ ω_{max} decreases getting far from ω_0

$\rightsquigarrow$ R_{max} decreases.