Midterm: Friday, Nov 16, in class

Topics: First order systems

- repeated eigenvalue
- zero eigenvalue
- Fundamental matrix, Wronskian and e^{tA}
- Non-homogeneous systems

Second order ODEs

- 2^{nd} order single ODE $\longleftrightarrow$ 1^{st} order system
- General sol'n for homogeneous eqt and IVP
- Mass-spring system (unforced)
- Non-hom (forced) 2^{nd} order ODE
- Mass-spring system (forced)

From Webpage:

Week 5, Lecture 14

to Week 10, Lecture 26 (last class)

Nonlinear Systems

Nov 7

Lecture 27

$\vec{x}' = \vec{F}(t, \vec{x}(t))$ General form :

$$\vec{x}'(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} x(t) + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \to \text{A linear system}$$

Constant coeff : $\vec{x}' = A\vec{x} + \vec{g}$

$\det A \neq 0 \longrightarrow (0,0)$ the only critical point.

For nonlinear system, things are much more complicated.

usually cannot be solved analytically :

Ex. $\begin{cases} x' = \sin x + y \\ y' = y(1-y) = y - y^2 \end{cases}$ or $\begin{cases} x' = 1 + xy + ty \\ y' = x - 2xy + t^2 \end{cases}$

nonlinear autonomous — nonlinear non-autonomous.

We consider autonomous nonlinear system :

$$\vec{x}' = \vec{F}(\vec{x}(t))$$

What we can do
- $\longrightarrow$ solve numerically (ODE 45)
- $\longrightarrow$ sketch the vector field and study the qualitative behaviour of sol'ns.
- $\longrightarrow$ study approximate linearized system.

Definition: An equilibrium solution (critical point) of the system $\vec{x}' = \vec{F}(\vec{x}(t))$ is a point $\vec{X}_0 = (x_0, y_0)$ such that $\vec{F}((x_0, y_0)) = 0 = \vec{F}(\vec{X}_0)$

For example for a 2×2 system:

$$\begin{cases} dx/dt = f(x,y) \\ dy/dt = g(x,y) \end{cases}$$

Here : $\vec{F}(\vec{x}(t)) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$

then $\vec{X}_0 = (x_0, y_0)$ is an equilibrium if

$$f(x_0, y_0) = 0 = g(x_0, y_0)$$

Compare this with $\vec{X}' = A\vec{X} \xrightarrow{\det A \neq 0}$ only $(0,0)$ critical point

but non linear systems can have several (many) critical points:

Ex 1: Let $x(t)$ and $y(t)$ be the population sizes of two species that compete for a shared resource.

Suppose in the absence of y:

the model for x is :

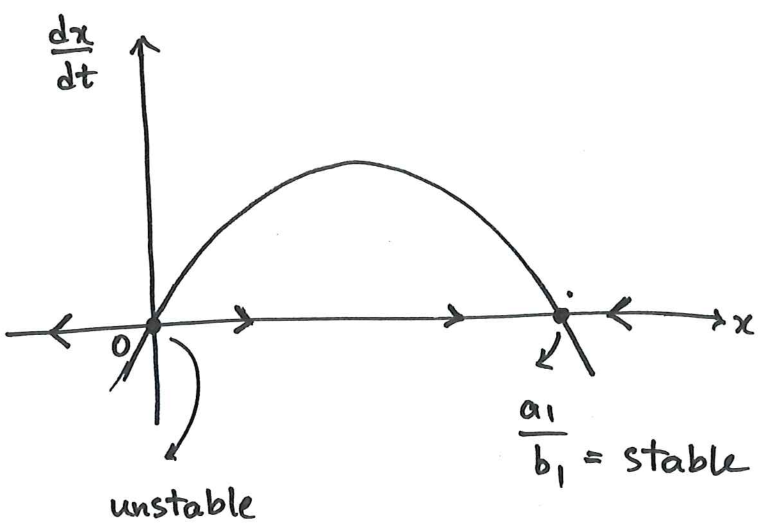
$$\frac{dx}{dt} = a_1 x - b_1 x^2 \qquad a_1, b_1, a_2, b_2 > 0$$

and in absence of x :

$$\frac{dy}{dt} = a_2 y - b_2 y^2$$

Chapter 1 :

Sketch $\frac{dx}{dt}$ vs. x :



$$a_1 x - b_1 x^2 = 0$$

$$x(a_1 - b_1 x) = 0$$

$$x = 0$$

$$x = \frac{a_1}{b_1}$$

when $y = 0$, $x = \frac{a_1}{b_1}$: stable

when $x = 0$, $y = \frac{a_2}{b_2}$: stable

Now consider the interaction between two species by adding a competition term :

$$\begin{cases} dx/dt = a_1 x - b_1 x^2 - c_1 xy \\ dy/dt = a_2 y - b_2 y^2 - c_2 xy \end{cases}$$

Critical points when : $\frac{dx}{dt} = 0 = \frac{dy}{dt}$

$$\begin{cases} x(a_1 - b_1 x - c_1 y) = 0 \\ y(a_2 - b_2 y - c_2 x) = 0 \end{cases}$$

(I) $x_0 = 0$, $y_0 = 0 \Rightarrow \vec{x}_0 = (0,0)$

(II) $x_0 = \frac{a_1}{b_1}$, $y_0 = 0 \Rightarrow \vec{x}_0 = (\frac{a_1}{b_1}, 0)$

(III) $x_0 = 0$, $y_0 = \frac{a_2}{b_2} \Rightarrow \vec{x}_0 = (0, \frac{a_2}{b_2})$

(IV) $a_1 - b_1 x - c_1 y = 0 \Rightarrow y = \frac{a_1 - b_1 x}{c_1}$

plug this into the 2^{nd} eqt & solve for x to get :

$$x_0 = \frac{a_2 c_1 - b_2 a_1}{c_1 c_2 - b_1 b_2}$$

and

$$y_0 = \frac{a_1 c_2 - a_2 b_1}{c_1 c_2 - b_1 b_2}$$

* $\vec{F}(\vec{x}(t))$ has 4 equilibria.

<u>Goal</u> : Determine the behaviour of the system near each equilibrium and study their stability.

We apply $\boxed{\text{Linearization}}$ to approximate the nonlin system with a linear one.

Recall :
Calculus : $f(x)$: linear approx around $x = x_0$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

if f is a function of two variables : $f(x,y)$
linearize around (x_0, y_0)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

where $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$

We'd like to approximate nonlinear system around each of its critical point with a linearization :

$$(x_0, y_0) \xrightarrow[\text{point}]{\text{critical}} f(x_0,y_0) = 0$$

$$\text{and } g(x_0,y_0) = 0$$

We have a 2D system $\begin{cases} dx/dt = f(x,y) \\ dy/dt = g(x,y) \end{cases}$

linearize f and g around (x_0, y_0): the critical point

$$\circledast \begin{cases} \dfrac{dx}{dt} = f(x,y) \approx \overset{0}{\cancel{f(x_0,y_0)}} + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\[2ex] \dfrac{dy}{dt} = g(x,y) \approx \overset{0}{\cancel{g(x_0,y_0)}} + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{cases}$$

Define $x - x_0 = u \Rightarrow \dfrac{dx}{dt} = \dfrac{du}{dt}$

$y - y_0 = v \Rightarrow \dfrac{dy}{dt} = \dfrac{dv}{dt}$

Rewrite $\circledast$ in terms of u and v:

$$\boxed{\begin{cases} du/dt = f_x(x_0,y_0)\,u + f_y(x_0,y_0)\,v \\[2ex] dv/dt = g_x(x_0,y_0)\,u + g_y(x_0,y_0)\,v \end{cases}}$$

This is a linear system. (linearized system for the original nonlinear one.)

In matrix form:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix}}_{\text{Jacobian matrix of } F:\ DF \text{ or } J} \begin{pmatrix} u \\ v \end{pmatrix}$$

In general:

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_n) \longmapsto \begin{pmatrix} F_1(x_1, \dots, x_n) \\ F_2(x_1, \dots, x_n) \\ \vdots \\ F_n(x_1, \dots, x_n) \end{pmatrix}$$

then

$$DF = J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

Jacobian matrix

Ex 1 : linearize around each critical point:

assume $a_1 = 1$ $\quad b_1 = 1$ $\quad c_1 = 1$

$\quad\quad\quad a_2 = 3$ $\quad b_2 = 4$ $\quad c_2 = 1$

$$DF = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{pmatrix}$$

Cont'd: Next Class