Nov 9
Lecture 28

Last Class:

Nonlinear autonomous systems:

$$\vec{x}' = \vec{F}(\vec{x}(t))$$

Equilibrium sol'n: $\vec{x}_o = (x_o, y_o)$ such that

$$\vec{x}' = \vec{F}(\vec{x}_o) = 0$$

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \overset{\text{Equilib}}{\Rightarrow} \quad f(x_o,y_o) = 0 = g(x_o,y_o)$$

Linearization:

around each critical point:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{(x_o,y_o)} \begin{pmatrix} u \\ v \end{pmatrix}$$

In general: $\boxed{\vec{U}' = D\vec{F}(\vec{x}_o)\,\vec{U}}$ → linear equation corresponding to the nonlinear system near the equilibrium $\vec{x}_o$.

Go back to the competing species example:

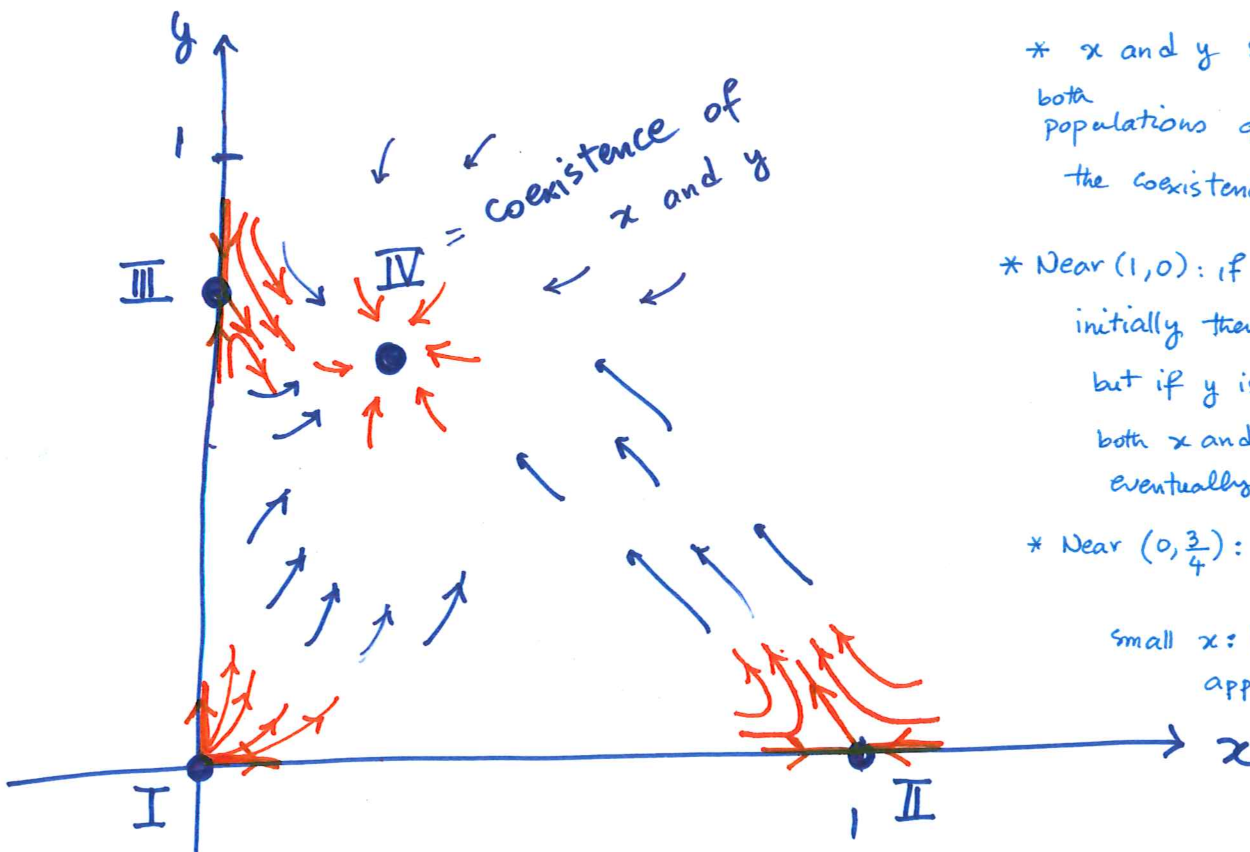
1

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy = f(x,y) \\[2ex] \dfrac{dy}{dt} = 3y - 4y^2 - xy = g(x,y) \end{cases}$$

Critical points:

$$\left.\begin{array}{ll} (\mathrm{I}) & \vec{X}_0 = (x_0, y_0) = (0,0) \\[2ex] (\mathrm{II}) & \vec{X}_0 = \left(\frac{a_1}{b_1}, 0\right) = (1,0) \\[2ex] (\mathrm{III}) & \vec{X}_0 = \left(0, \frac{a_2}{b_2}\right) = \left(0, \frac{3}{4}\right) \end{array}\right\} \text{absence of one or both species.}$$

(IV) $\vec{X}_0 = (\ldots, \ldots) = \left(\frac{1}{3}, \frac{2}{3}\right) \rightarrow$ presence of both species.



* x and y small (near origin) both populations grow to the coexistence equilibrium.

* Near (1,0): if NO y exists initially then it remains 0 but if y is small initially both x and y approach IV eventually.

* Near $(0, \frac{3}{4})$: NO x initially $\Rightarrow$ NO x forever

small x: both x and y approach IV

* Check the next page for this diagram

Linearize around each critical point:

$$DF = \begin{pmatrix} \overset{f_x}{1-2x-y} & \overset{f_y}{-x} \\ \underset{g_x}{-y} & \underset{g_y}{3-8y-x} \end{pmatrix}$$

(I) $(0,0)$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

eigval: $r_1 = 1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$r_2 = 3 \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ Source node unstable.

(*) Note that DF is invertible for each critical point:

$\det(DF(x_0)) \neq 0$

So each linear system has only 1 critical point.

$\Rightarrow$ This is called an isolated critical point

(II) $(1,0)$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow r_1 = 2$; $\vec{v}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$r_2 = -1$; $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ Saddle unstable

(III): $(0, \frac{3}{4})$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ -3/4 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow r_1 = \frac{1}{4}$; $\vec{v}_1 = \begin{pmatrix} 13/3 \\ -1 \end{pmatrix}$

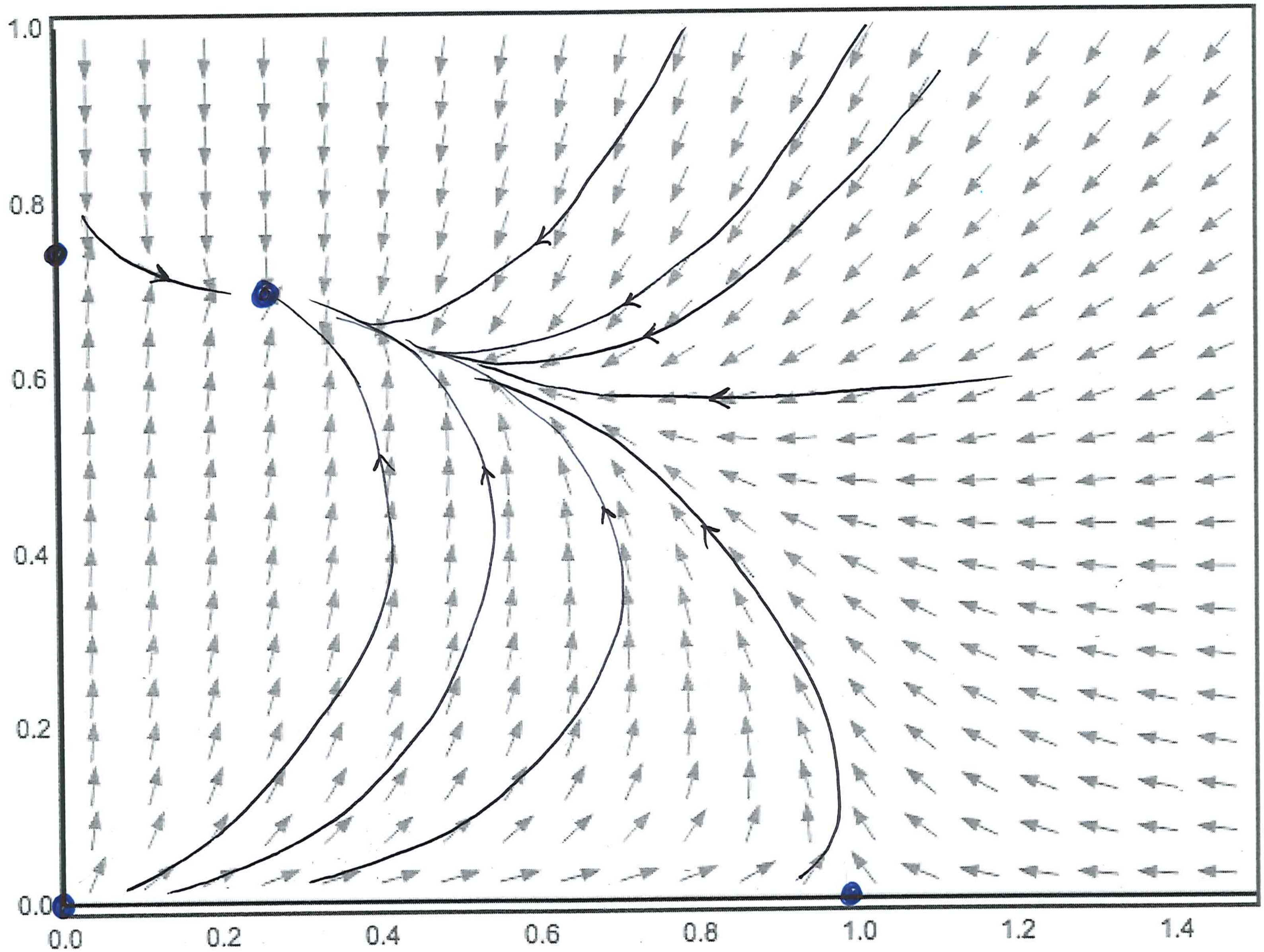
$r_2 = -3$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ Saddle unstable

(IV): $(\frac{1}{3}, \frac{2}{3})$: $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{8}{3} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

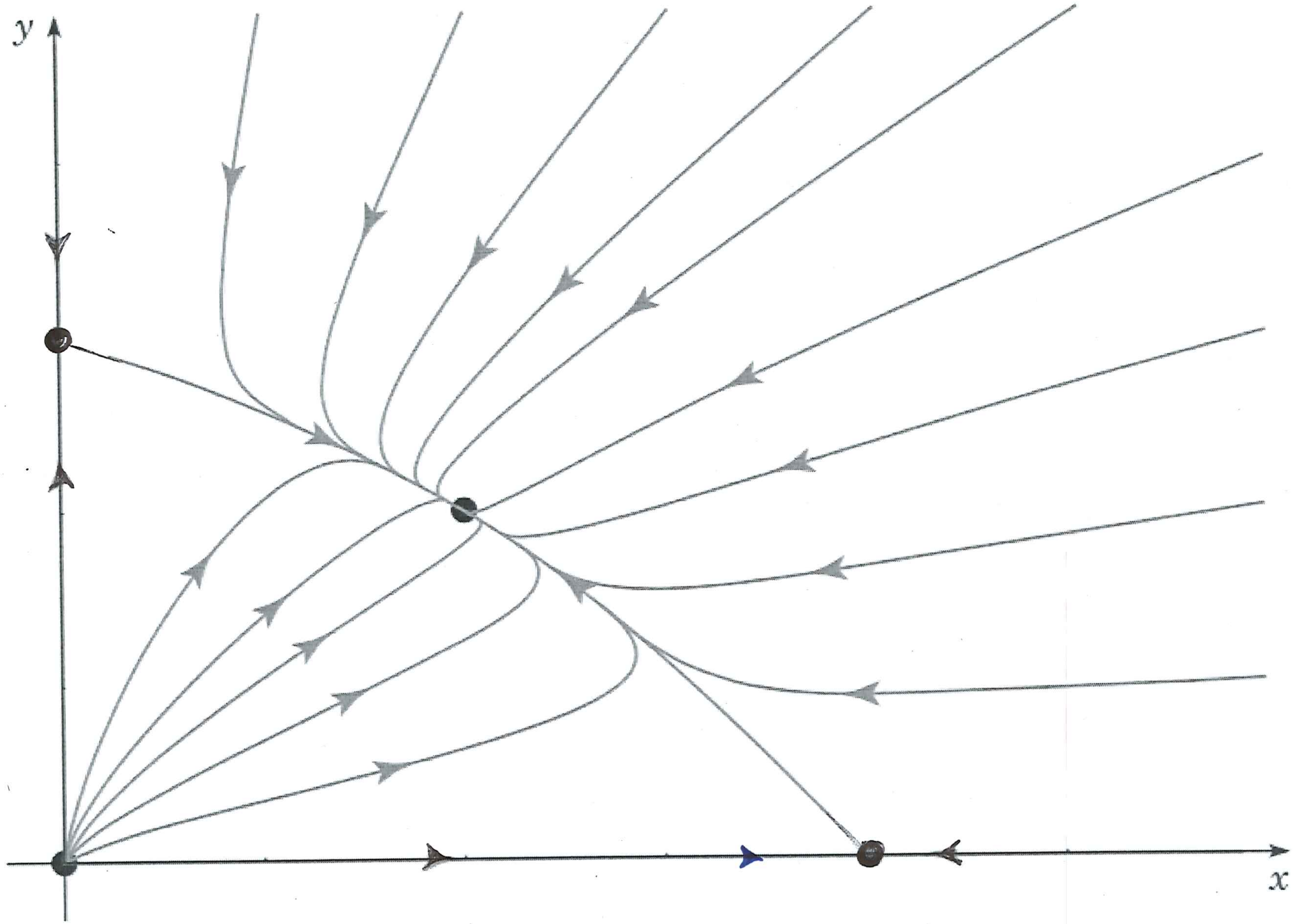
$r_{1,2} = \frac{1}{6}(-9 \pm \sqrt{57}) < 0 \Rightarrow$ asym. stable sink node

Direction field for the competing species.
+ some trajectories



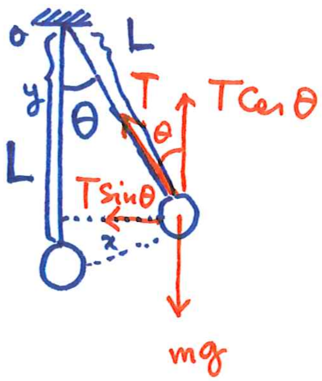
Phase Portrait of the competing species system

* The nonlinear system behaves similar to its linearization.



Ex 2: Motion of the pendulum:

Model the angular position of the pendulum over time: $\theta(t)$



$$\begin{cases} F_x = -T\sin\theta = m x'' \\ F_y = T\cos\theta - mg = m y'' \end{cases}$$

also we know:

$$\begin{cases} x = L\sin\theta \\ y = -L\cos\theta \end{cases}$$

Some algebra gives

The model: $L\theta'' + g\sin\theta = 0$ undamped

with damping: $L\theta'' + c\theta' + g\underline{\sin\theta} = 0$

nonlinear 2^{nd} order ODE

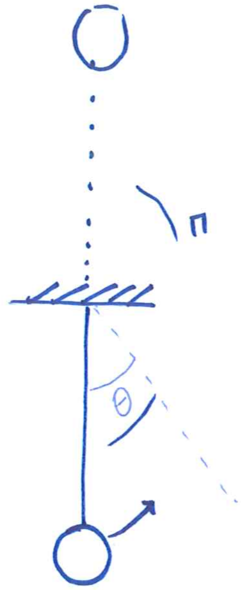
Make it a system

$\theta = x \quad \Rightarrow x' = \theta' = y$

$\theta' = y \quad \Rightarrow \quad \theta'' = y' = -\frac{c}{L}\theta' - \frac{g}{L}\sin\theta$

$$= -\frac{c}{L}y - \frac{g}{L}\sin x$$

$$\begin{cases} x' = y \rightsquigarrow f(x,y) \\ y' = -\frac{c}{L}y - \frac{g}{L}\sin x \rightsquigarrow g(x,y) \end{cases} \xRightarrow[\text{points:}]{\text{Critical}} \begin{array}{l} y = 0 \quad (\theta' = 0) \\ \sin x = 0 \\ \Rightarrow \underset{\theta}{x} = n\pi \quad n = 0, \pm 1, \pm 2, \dots \end{array}$$



we expect to have:

down state $\leftsquigarrow (2n\pi, 0)$ to be stable.

and

up state $\leftsquigarrow ((2n+1)\pi, 0)$ unstable

linearize the nonlinear system:

$$DF = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos\theta & -\frac{c}{L} \end{pmatrix} \Longrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = DF(\vec{X}_0)\begin{pmatrix} u \\ v \end{pmatrix}$$

(I) around down: $(2n\pi, 0)$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{c}{L} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

eigval:

$$r^2 + \frac{c}{L} r + \frac{g}{L} = 0$$

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4gL}}{2L}$$

α $\quad -\beta$

$\rightarrow \alpha = \frac{c}{2L}$

$\beta = \frac{\sqrt{4gL - c^2}}{2L}$

two complex with negative real part $\Rightarrow$ spiral sink

$\Rightarrow r_{1,2} = -\alpha \pm i\beta$

(II): around up $((2n+1)\pi, 0)$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{c}{L} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

eigval:

$$r^2 + \frac{c}{L} r - \frac{g}{L} = 0$$

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 + 4gL}}{2L}$$

$c^2 + 4gL > 0$

$r_1^+ > 0$

$r_2^- < 0$

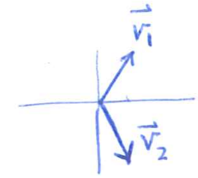
$\Rightarrow$ so $((2n+1)\pi, 0)$ is a saddle.

For case II : Sol'n:

$$\begin{pmatrix}\theta(t)\\ \theta'(t)\end{pmatrix} = c_1\begin{pmatrix}1\\ \underset{>0}{r_1}\end{pmatrix} e^{r_1 t} + c_2\begin{pmatrix}1\\ \underset{<0}{r_2}\end{pmatrix} e^{r_2 t}$$

$\vec{v}_1 = \begin{pmatrix}1\\ r_1\end{pmatrix}$ $\quad$ $\vec{v}_2 = \begin{pmatrix}1\\ r_2\end{pmatrix}$

$r_1 > 0$ $\quad$ $r_2 < 0$

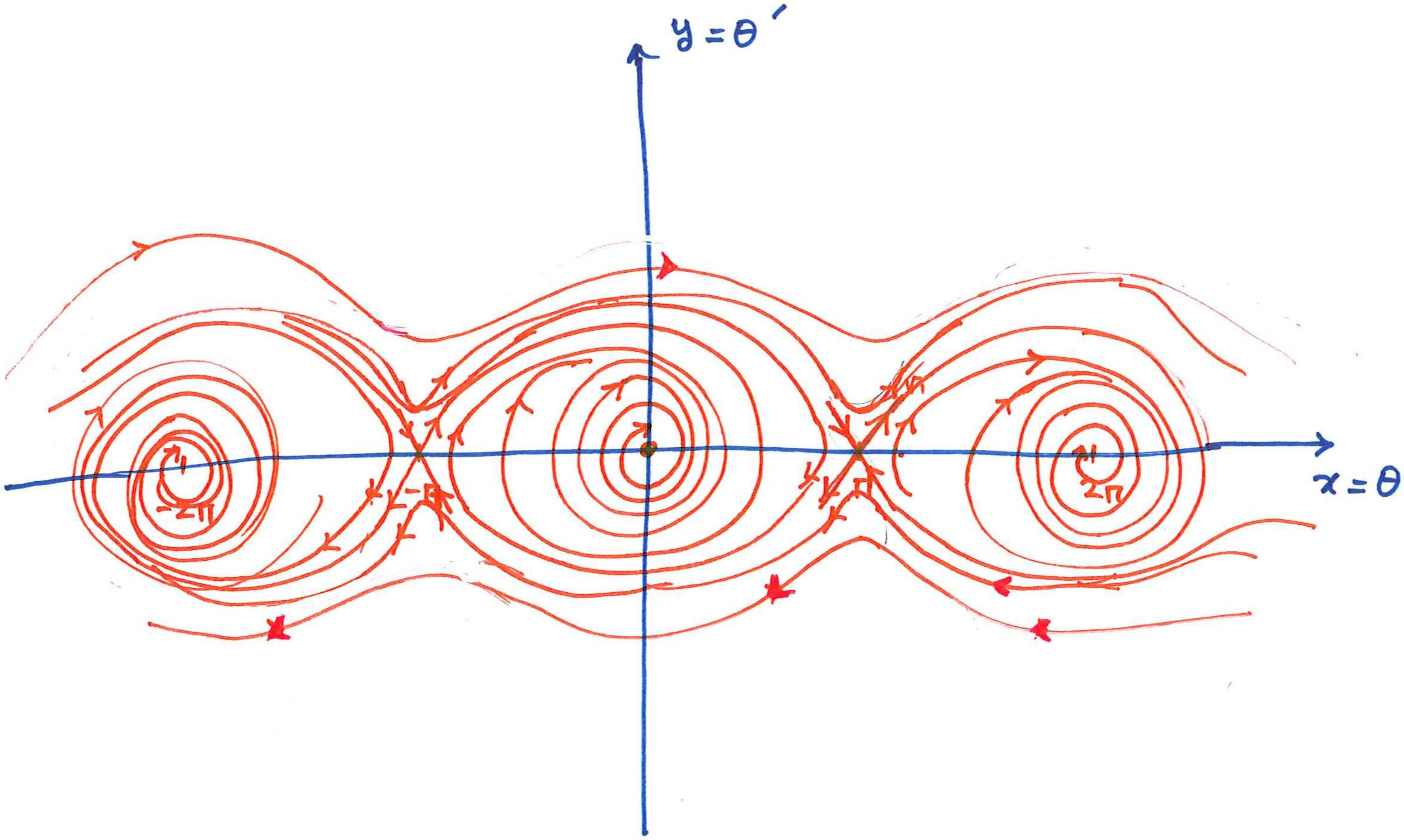


For case I :

if $\boxed{c^2 - 4gL < 0}$ $\Rightarrow$ complex eigval with negative real part

$$\theta(t_1) = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t$$

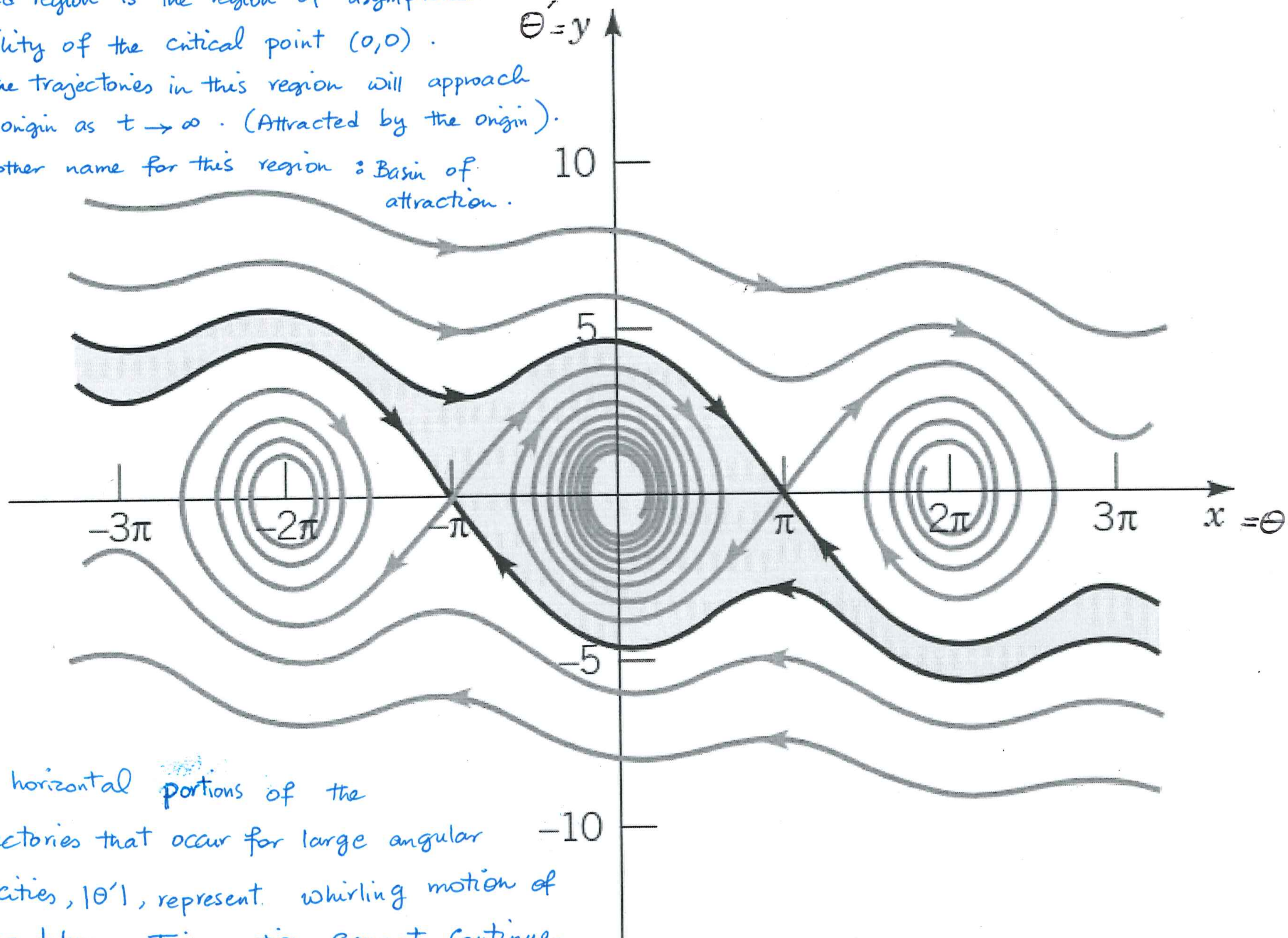
under damping .



Phase portrait for damped pendulum (light damping ⇝ underdamping)

* Shaded region is the region of asymptotic stability of the critical point (0,0). All the trajectories in this region will approach the origin as $t \to \infty$. (Attracted by the origin).

→ Another name for this region : Basin of attraction.



* Wavy horizontal portions of the trajectories that occur for large angular velocities, $|\theta'|$, represent whirling motion of the pendulum. This motion cannot continue indefinitely since eventually due to damping pendulum oscillates around its downward position.