Pendulum Motion Cont'd: Nov 19, Lecture 30

Linear system around down-state equilibria

As we saw, up-state equilib. are always saddles.

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{c}{L} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$r_{1,2} = \frac{-c}{2L} \pm \frac{\sqrt{c^2-4gL}}{2L}$$

(I) $c^2 - 4gL < 0$ → light damping (under damped)

$$r_{1,2} = -\frac{c}{2L} \pm i\,\frac{\sqrt{4gL - c^2}}{2L} = \alpha \pm i\beta$$

$$\theta(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

⇒ Oscillatory motion with decreasing amplitude.

⇒ spiral sink

(II) $c^2 - 4gL > 0$ → Strong damping (over-damped)

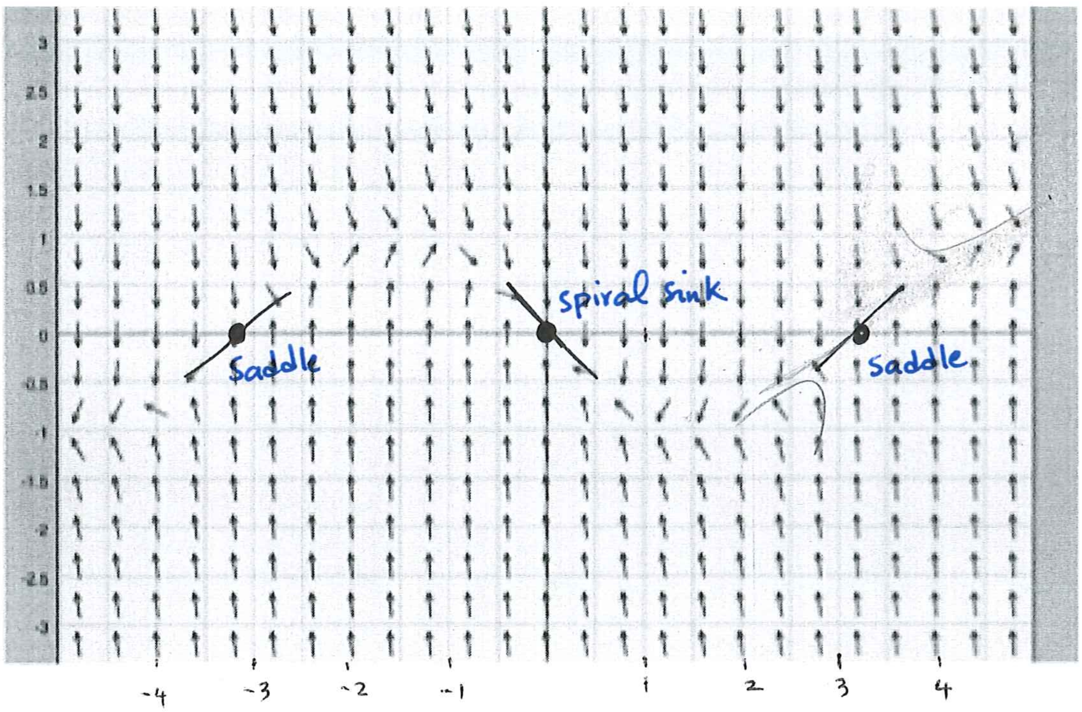
r_1, r_2 both real and negative

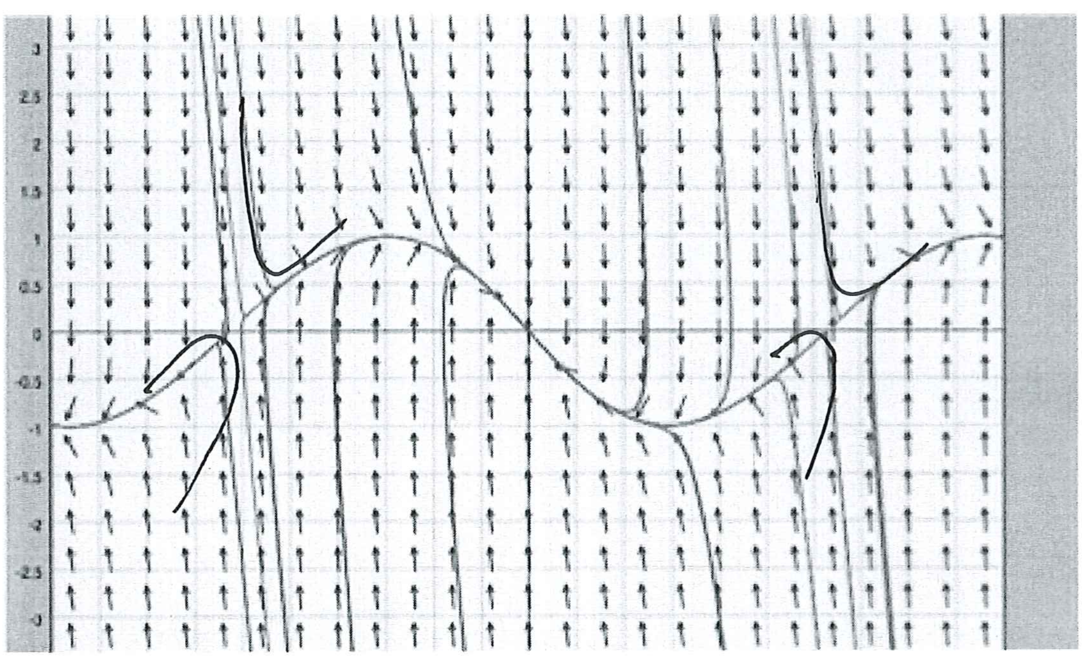
$$\theta(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

⇒ fast decay ⇒ sink node

Over-damped pendulum

direction field and some trajectories

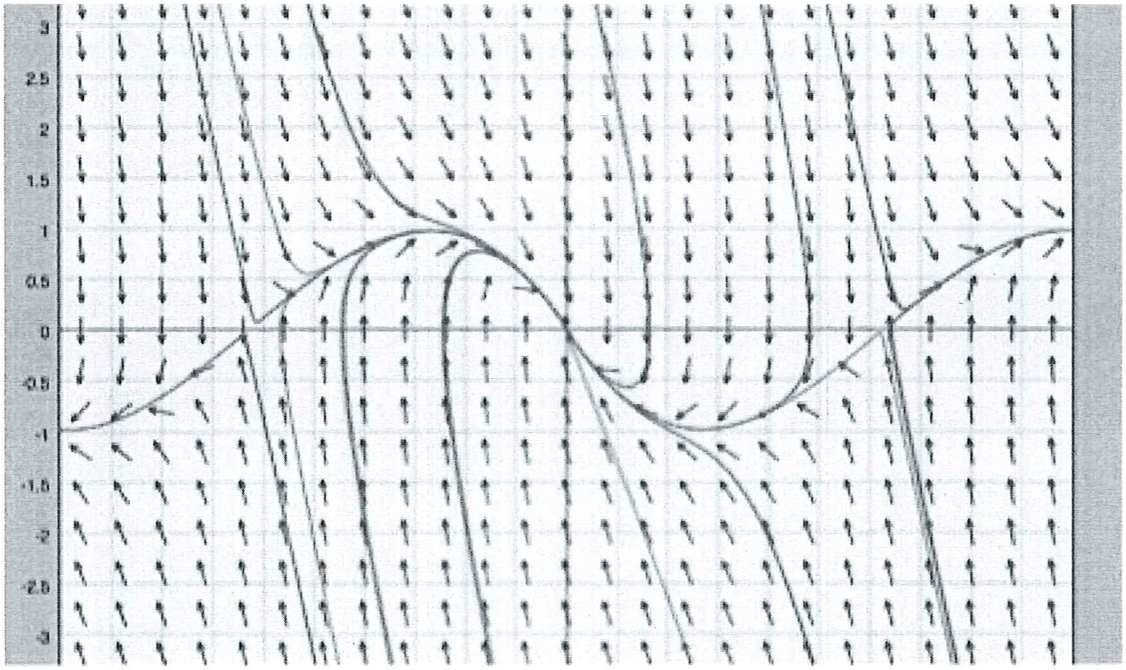




Critically damped pendulum :

III . $c^2 - 4gL = 0 \Rightarrow r_1 = r_2 = -\frac{c}{2L} = r$

$\theta(t) = c_1 e^{rt} + c_2 t e^{rt}$ $\rightsquigarrow$ asym. stable (improper node)



Exercise : Assume that the system described by a differential equation for a mass-spring or a pendulum, is critically damped or overdamped. Prove that the mass or the pendulum can pass through the equilibrium position at most once, regardless of the initial condition.

Undamped Pendulum:

$$L\theta'' + g\sin\theta = 0 \quad \xrightarrow{\text{equilibria}} \quad \begin{array}{l} \theta = n\pi \\ \theta' = 0 \end{array} \begin{array}{l} \nearrow (2n\pi, 0) \quad \text{down} \\ \searrow ((2n+1)n\pi, 0) \quad \text{up} \to \text{still saddle} \end{array}$$

$\theta = x$, $\theta' = y$ $\xrightarrow{\text{system}}$ and linearize around down-state equilibria:

Recall: As a system:

$\theta = x \Rightarrow \theta' = x' = y$

$\theta' = y \Rightarrow \theta'' = -\frac{g}{L}\sin\theta = -\frac{g}{L}\sin x$

$\Rightarrow DF = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos\theta & 0 \end{pmatrix}$

$DF((2n\pi, 0)) = \begin{pmatrix} 0 & 1 \\ -g/L & 0 \end{pmatrix}$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix} \vec{x}$$

$$\Rightarrow r^2 + \frac{g}{L} = 0$$

$$\Rightarrow r = \pm i\sqrt{\frac{g}{L}} = \pm i\omega_0$$

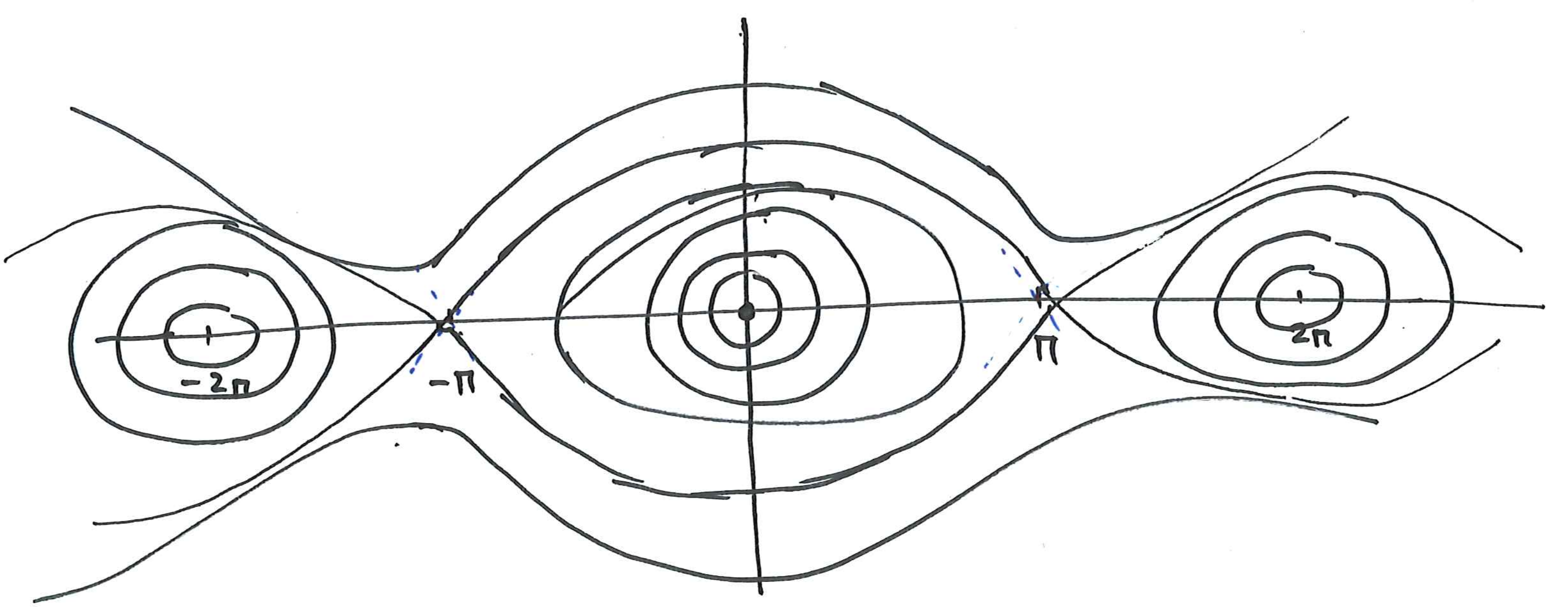
$$\theta(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

$\Rightarrow$ pure oscillation ; fixed amplitude

As before:

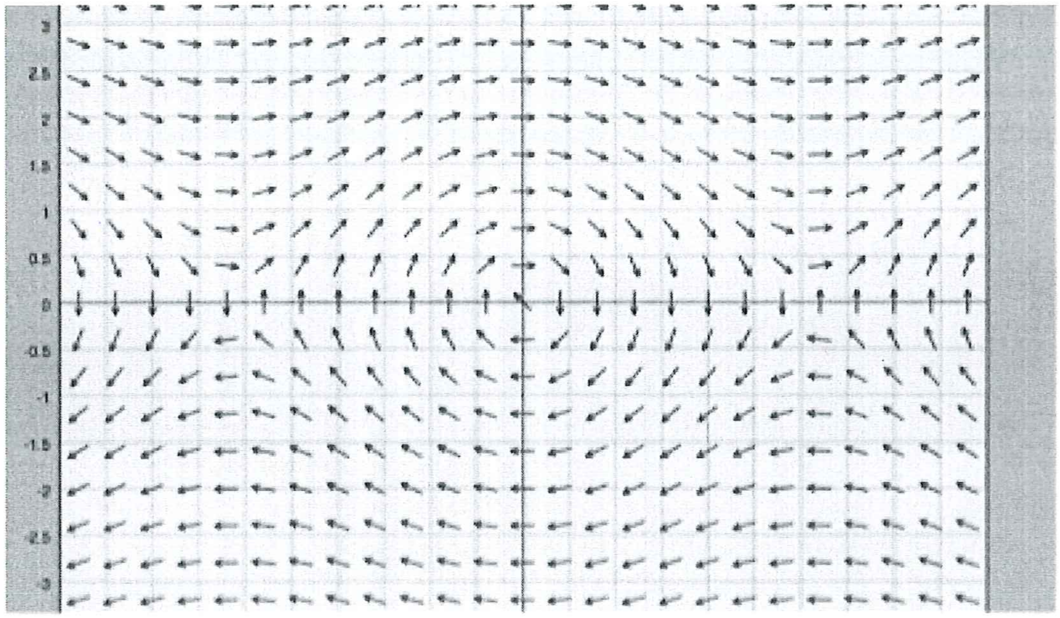
up-state equilibria : saddle unstable

down-state " : centre stable (<u>not</u> asym. stable)



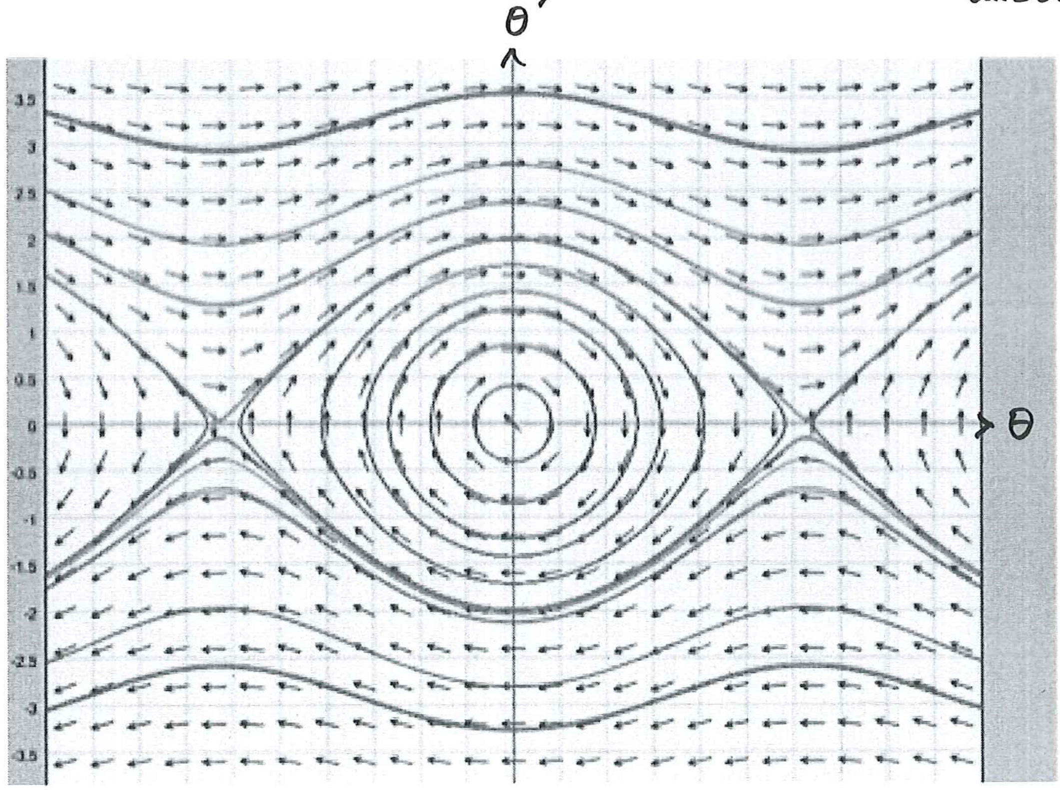
Undamped pendulum :

$L\theta'' + g\sin\theta = 0$



* for small θ : elliptical orbits ⇝ indefinite oscillation

* large θ and θ' : no longer elliptical orbits ⇝ motion becomes unbounded.



eigenvalues of Jacobian	stability of equilibria for linear system $\vec{x}' = D\vec{F}(x_0)\vec{x}(t)$	type of equilibria	stability for non-linear system $\vec{x}' = \vec{F}(\vec{x}(t))$	type of equilibria
distinct: $r_1, r_2 > 0$	unstable	source node	unstable	source node
distinct: $r_1, r_2 < 0$	asym. stable	sink	asym. stable	sink
distinct: $r_1 < 0 < r_2$	unstable	saddle	unstable	saddle
$0 \neq r_1 = r_2 > 0$	unstable	improper/proper node	unstable	improp/prop. or sink or source
$0 \neq r_1 = r_2 < 0$	asym. stable	" "	asym. stable	improp/prop.
$r_{1,2} = \alpha \pm i\beta$				
$\alpha > 0$	unstable	spiral source	unstable	spiral source
$\alpha < 0$	asym. stable	spiral sink	asym. stable	spiral sink
$\alpha = 0$	stable	centre	Indeterminate	Indeterminate Center or Spiral

(*) If $r_1 = r_2$ in the linear system, it's possible that the nonlinear terms affect the system so that $r_1 \neq r_2$.

then the type of the equilibria changes from improper/proper node to a sink or source node, but the asym. stability or unstability of the equilibria will not change.

(*) If $r_{1,2} = \alpha \pm i\beta$ and $\alpha = 0$ for the linearized system, a small nonlinear term may change a centre to a spiral (asym. stable or unstable)

so the behaviour of non-linear around these equilibria is indeterminate.

Except these two cases, the type and stability of the equilibria of the nonlinear system is similar to the simpler linearized version $\Rightarrow$ The nonlinear system is in fact almost linear.

Example 3. Predator – Prey Equation :

Assumptions :

(1) In the absence of the predator, the prey grows at a rate proportional to the current population

$$\Rightarrow \quad dx/dt = ax, \quad a>0 \quad \text{when } y=0$$

(2) In the absense of the prey, the predator dies out

$$\Rightarrow \quad dy/dt = -cy, \quad c>0 \quad \text{when } x=0$$

(3) encounters between predator and prey is proportional to the product of their population. Encounters promote the growth of predator and prevent the growth of the prey, thus the model is

$$\begin{cases} dx/dt = ax - \alpha xy \\ dy/dt = -cy + \gamma xy \end{cases} \qquad a, c, \alpha, \gamma > 0$$

Exercise for you : Analyze the qualitative behaviour of the system.

- critical points
- linearization
- trajectories

Simpler case : $a = 1$, $\alpha = 0.5$, $c = 0.75$, $\gamma = 0.25$