Laplace Transform:

Nov 21
Lecture 31

Given a function $f(t)$, its Laplace Transform (LT) of f is an integral transform of the form:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

Recall: Improper integral over an unbounded interval.

$$\int_a^{\infty} f(t)\, dt = \lim_{b \to \infty} \int_a^{b} f(t)\, dt$$

If the limit exists and it is equal to some finite L then we say that the improper integral converges to L.

Example. Determine for what values of c, the integral

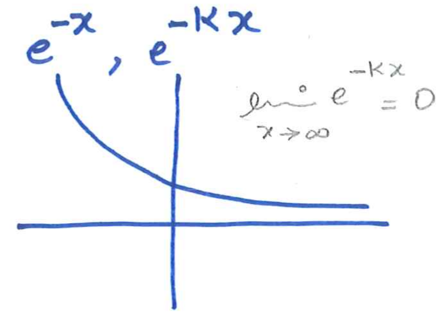
$$\int_0^{\infty} e^{ct}\, dt \quad \text{converges}.$$

$$\int_0^{\infty} e^{ct}\, dt = \lim_{b \to \infty} \int_0^{b} e^{ct}\, dt = \lim_{b \to \infty} \frac{1}{c} e^{ct} \Big|_0^b$$

$$= \lim_{b \to \infty} \frac{1}{c} e^{bc} - \frac{1}{c}$$

if $c > 0$: $\lim_{b \to \infty} e^{bc} = \infty$

$\Rightarrow$ integral diverges.



if $c < 0$: $\lim_{b \to \infty} e^{bc} = 0 \Rightarrow$ integral converges to $-\frac{1}{c}$

if $c = 0$: $\int_0^\infty dt = \lim_{b \to \infty} \int_0^b dt = \lim_{b \to \infty} t \Big|_0^b = \infty \Rightarrow$ diverges

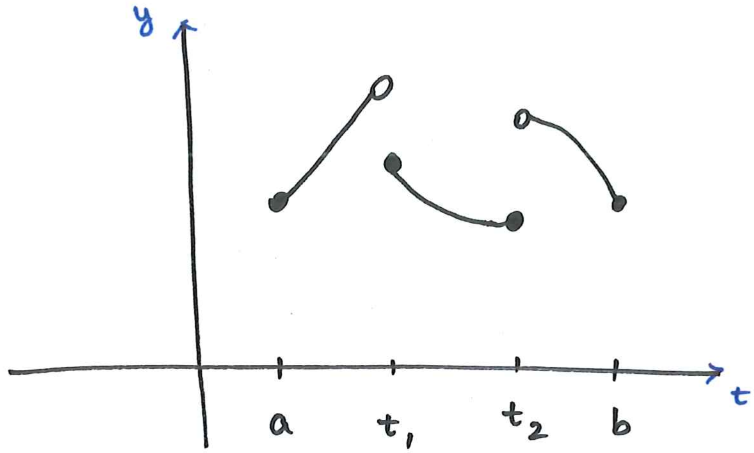
So $\int_0^\infty e^{ct}\, dt$ converges when $c < 0$.

<u>Question</u> : When does $\mathcal{L}\{f(t)\}$ exist ?

<u>Definition</u> : A function f is called piecewise continuous on an interval $[a,b]$ if we can partition $[a,b]$ to a finite number of points:

$$t_0 = a < t_1 < t_2 < \dots < t_n = b$$

so that f is continuous on each subinterval (t_i, t_{i+1}) and the one-sided limits at each endpoint exist within the subinterval.



In other word, f has finite number of jump discontinuities.

* Fact: piecewise functions are always integrable.

Existence of LT

Theorem: Suppose

(1) f is piece-wise continuous on the interval $[0,b]$ for any positive b.

(2) There exist constants α, K, M $(K, M > 0)$ such that

$$|f(t)| \leq K e^{\alpha t} \quad \text{for } t \geq M$$

Then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > \alpha$.

Idea of the proof:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Laplace variable

$$\int_0^\infty e^{-st} f(t)\, dt = \underbrace{\int_0^M e^{-st} f(t)\, dt}_{\text{Condition (1) implies that this is finite.}} + \int_M^\infty e^{-st} f(t)\, dt$$

$$\int_M^\infty e^{-st} f(t)\, dt \le \int_M^\infty e^{-st} |f(t)|\, dt$$

$$\underset{(2)}{\overset{\text{Condition}}{\le}} \int_M^\infty e^{-st} K e^{+\alpha t}\, dt$$

$$= K \int_M^\infty e^{(\alpha - s)t}\, dt$$

The last integral converges when $\alpha - s < 0$

Comparison test for integral implies $L\{f(t)\}$ exists when $\boxed{s > \alpha}$ $\square$.

The interval on which LT exists is called the domain of LT.

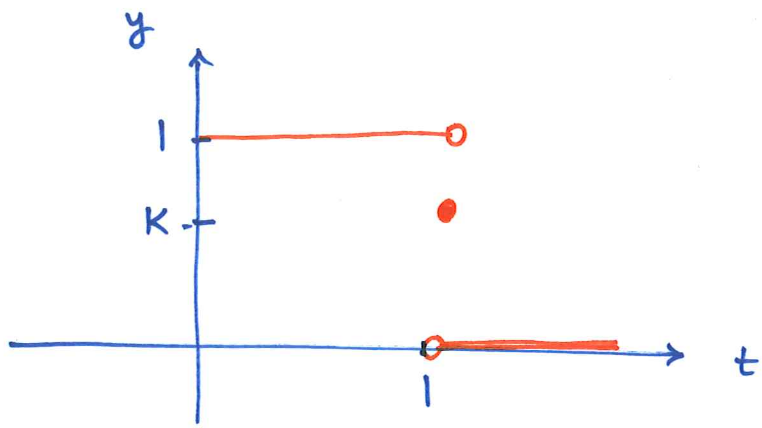
(*) Condition (2) implies that f can NOT grow faster that exp function. (f is of exponential order.)

for example: $f(t) = e^{t^2}$ is NOT of exponential order.

There exist NO K and NO α s.t.

$$e^{t^2} < K e^{\alpha t}$$

Ex 1 . Find LT of the function given by



$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ K & t = 1 \\ 0 & t > 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} \cdot 1 \, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^1$$

$$= -\frac{1}{s} e^{-s} + \frac{1}{s}$$

$$= \frac{1 - e^{-s}}{s} \qquad s > 0$$

(*) Constant K does NOT play any role.

Ex 2 : $\mathcal{L}\{1\} = ? = \frac{1}{s} \qquad s > 0$

$$\int_0^\infty e^{-st}\, dt = \lim_{b \to \infty} \int_0^b e^{-st} dt = \lim_{b \to \infty} -\frac{1}{s} e^{-st} \Big|_0^b = \frac{1}{s}$$

Ex 3: $\mathcal{L}\{e^{at}\} =$

$a - s < 0$

$s > a$

$$\int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{(a-s)t}\, dt$$

$$= \lim_{b \to \infty} \frac{1}{a-s} e^{(a-s)t} \Big|_0^b$$

$$= \frac{1}{s-a}$$

Ex 4: $\mathcal{L}\{\sin(at)\}$ and $\mathcal{L}\{\cos(at)\}$

Easy way: Euler's formula:

$$e^{iat} = \cos(at) + i \sin(at)$$

* **Remark**: LT is linear.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$$

Easy to prove (integrals are linear)

This is just a notation for $\cos + i\sin$

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat}\, dt \qquad s>0$$

$$= \lim_{b\to\infty} \int_0^b e^{(ia-s)t}\, dt$$

$$= \lim_{b\to\infty} \frac{1}{ia-s} e^{(ia-s)t} \Big|_0^b$$

$$* \lim_{b\to\infty} e^{(ia-s)b} = 0$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} + i\,\mathcal{L}\{\sin(at)\} = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

* You could apply LT on each function separately and do integration by parts twice to get $F(s)$.