Nov 23
Lecture 32

Last Class:

Laplace Transform (LT)

Definition: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Existence of LT:

Theorem 1: If (1) f is piecewise cont's on $[0,b]$ for any positive b.

(2) $|f(t)| \leq K e^{\alpha t}$ $t \geq M$

f is of exponential order

for some α real
$K, M > 0$.

Then $L\{f\} = F(s)$ exists when $s > \alpha$.

LT of basic functions:

$\mathcal{L}\{1\} = \frac{1}{s}$ $s>0$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $s>a$

$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$ $s>0$

$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ $s>0$

Using LT to solve an initial value problem.

Suppose we want to solve

$$a y'' + b y' + c y = f(t) \quad (*)$$

with $y(0)$ and $y'(0)$ given.

we need:

→ LT of derivatives of y:

Assume y satisfies conditions of the Theorem 1:

We'd like to find $\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$:

$$\mathcal{L}\{y'\} = \int_0^\infty e^{-st} y'(t)\, dt = y e^{-st} \Big|_0^\infty + s \int_0^\infty e^{-st} y \, dt$$

$$\boxed{\begin{array}{ll} e^{-st} = u & y'\,dt = dv \\ -s e^{-st}\,dt = du & y = v \end{array}}$$

y is such that $\lim_{t\to\infty} y e^{-st} = 0$

$$= -y(0) + s\,\mathcal{L}\{y\}$$

$$\Rightarrow \boxed{\mathcal{L}\{y'\} = s\,Y(s) - y(0)}$$

$$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\} = s\,\mathcal{L}\{y'\} - y'(0)$$

$$= s\big(s\,\mathcal{L}\{y\} - y(0)\big) - y'(0)$$

$$= s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)$$

$$\Rightarrow \boxed{\mathcal{L}\{y''\} = s^2\, Y(s) - s\,y(0) - y'(0)}$$

In general:

$$\mathcal{L}\{y^{(n)}\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots$$
$$- s\, y^{(n-2)}(0) - y^{(n-1)}(0)$$

Apply LT to both sides of equation (*):

$$a\,\mathcal{L}\{y''\} + b\,\mathcal{L}\{y'\} + c\,\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\underline{as^2 Y(s)} - asy(0) - ay'(0) + \underline{bs\,Y(s)} - by(0) + \underline{c\,Y(s)}$$
$$= F(s)$$

$$Y(s)\left(as^2 + bs + c\right) = asy(0) + ay'(0) + by(0) + F(s)$$

$$\Rightarrow \quad Y(s) = \frac{(as+b)\,y(0) + a\,y'(0) + F(s)}{as^2 + bs + C}$$

(1) initial conditions are already included.

(2) algebraic equation, no longer a differential eqt.

(3) non-homogeneous term is already included $\rightsquigarrow$ no need to worry about y_p.

(4) works for higher order ODEs.

The difficulty with LT in an ODE is that we need to find a function $y(t)$ whose LT is $Y(s)$.

$\Rightarrow$ We need to use inverse of $\mathcal{L}$T.

One can show that the mapping

$$f(t) \longmapsto \mathcal{L}\{f(t)\}$$

is one-to-one (on a set of appropriate functions)

so we can invert this map:

$$\mathcal{L}\{y\} = Y(s) \xrightarrow{\mathcal{L}^{-1}} \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\{\mathcal{L}\{y(t)\}\}$$

$$= y(t)$$

<u>Example</u>. Solve IVP

$$y'' + y = \sin(2t) \qquad \text{using LT.}$$
$$y(0) = 2, \quad y'(0) = 1$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

$$\underbrace{s^2 Y(s) - s\overset{2}{\cancel{y(0)}} - \overset{1}{\cancel{y'(0)}}} + \underbrace{Y(s)} = \frac{2}{s^2+4}$$

$$(s^2+1)\,Y(s) = 2s+1 + \frac{2}{s^2+4}$$

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

- $$\frac{2s+1}{s^2+1} = 2\,\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$\mathcal{L}^{-1}$ $\quad$ $\mathcal{L}^{-1}$

$$= 2\cos(t) + \sin t$$

- $$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$= \frac{As^3+4As+Bs^2+4B+Cs^3+Ds^2+Cs+D}{(s^2+1)(s^2+4)}$$

$$\left.\begin{aligned} A+C&=0\\ B+D&=0\\ 4A+C&=0\\ 4B+D&=2 \end{aligned}\right\} \Rightarrow \begin{aligned} &A=0=C\\ &B=2/3,\quad D=-2/3 \end{aligned}$$

$$= \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4}$$

$$" \frac{1}{3}\cdot\frac{2}{s^2+4}$$

$$= \frac{2}{3} \sin t - \frac{1}{3} \sin(2t)$$

Collect all terms:

$$\Rightarrow \quad y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t) \quad \checkmark$$

Given $f(t)$:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a)$$

Ex. $\mathcal{L}\{\underbrace{e^{2t}}_{F(s-2)} \underbrace{\sin t}_{F(s)=\frac{1}{s^2+1}}\} = \dfrac{1}{(s-2)^2+1}$

Ex. $\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s+2)^2+9}\right\} =$

$$= \frac{s+2-3}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{3}{(s+2)^2+9}$$

$$= e^{-2t}\underbrace{\cos(3t)}_{\frac{s}{s^2+9}} - e^{-2t}\underbrace{\sin(3t)}_{\frac{3}{s^2+9}}$$

Ex: Solve $y'' + 4y' + 5y = e^t$

$$y(0) = 1 \quad , \quad y'(0) = 2$$

$$Y(s) = \frac{s+6}{s^2+4s+5} + \frac{1}{(s-1)(s^2+4s+5)}$$

Complete the square

$$s^2+4s+4+1$$
$$= (s+2)^2 + 1$$

partial fraction

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+4s+5}$$

Solution : $\mathcal{L}\{y''\} + 4\,\mathcal{L}\{y'\} + 5\,\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$

$$s^2 Y(s) - s\,\overset{1}{\cancel{y(0)}} - \overset{2}{\cancel{y'(0)}} + 4s\,Y(s) - 4\,\overset{1}{\cancel{y(0)}} + 5\,Y(s) = \frac{1}{s-1}$$

$$\Rightarrow (s^2+4s+5)\,Y(s) = s+6 + \frac{1}{s-1}$$

$$\Rightarrow Y(s) = \frac{s+6}{s^2+4s+5} + \frac{1}{(s-1)(s^2+4s+5)}$$

$$= \frac{s+6}{(s+2)^2+1} + \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+5}$$

- Find A and B :

$$As^2 + 4As + 5A + Bs^2 + Cs - Bs - C = 1$$

$$\Rightarrow A+B=0 \quad , \quad 4A+C-B=0 \quad \Rightarrow \quad A=\frac{1}{10} \ , \ B=-\frac{1}{10} \ , \ C=-\frac{1}{2}$$
$$5A - C = 1$$

$$\Rightarrow \quad \frac{1}{(s-1)(s^2+4s+5)} = \frac{1/10}{s-1} - \frac{\frac{s}{10}+\frac{1}{2}}{(s+2)^2+1}$$

→ easier if multiply & divide by 10

- $$\mathcal{L}^{-1}\left\{\frac{1}{10}\cdot\frac{1}{s-1}\right\} = \frac{1}{10}e^{t}$$

- $$\mathcal{L}^{-1}\left\{\frac{1}{10}\frac{s+5}{(s+2)^2+1}\right\} = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} + 3\cdot\frac{1}{(s+2)^2+1}\right\}$$

$$= \frac{1}{10}e^{-2t}\cos t + \frac{3}{10}e^{-2t}\sin t$$

- $$\mathcal{L}^{-1}\left\{\frac{s+6}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1} + \frac{4}{(s+2)^2+1}\right\}$$

$$= e^{-2t}\cos t + 4e^{-2t}\sin t$$

$$\xrightarrow[\text{all terms}]{\text{Collect}} \quad \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{1}{10}e^{t} + \frac{9}{10}e^{-2t}\cos t + \frac{37}{10}e^{-2t}\sin 2t$$

Question: What if the given initial condition is NOT at $t=0$?

$y(t_0)$ and $y'(t_0)$?

Practice : (1) Solve

$$y^{(4)} - y = 0$$

$$y(0) = y''(0) = y'''(0) = 0 \quad , \quad y'(0) = 1$$

(2) Find the inverse LT of :

(a) $F(s) = \dfrac{2s + 2}{s^2 + 2s + 5}$

(b) $F(s) = \dfrac{2s - 3}{s^2 - 4}$

(c) $F(s) = \dfrac{8s^2 + 4s + 12}{s(s^2 + 4)}$

(d) $F(s) = \dfrac{1 - 2s}{s^2 + 4s + 5}$