Last Week Summary :

Nov 26
Lecture 33

Laplace Transform of $f(t)$: $\mathcal{L}\{f(t)\}$

$$= \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

↳ exists when (1) f is pw cont's
(2) f is of exponential order.

$|f(t)| \le K e^{\alpha t}$

Elementary LTs :

- $\mathcal{L}\{1\} = \dfrac{1}{s} \quad s>0$
- $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} \quad s>a$
- $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$, $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2} \quad s>0$
- $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad s > a+\alpha$

  multiply f by e^{at} ⇝ shift F "a" units
- Example : $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$

  $\mathcal{L}\{e^{at}\cos(bt)\} = \dfrac{(s-a)}{(s-a)^2+b^2}$

LT of derivatives : Assume $\mathcal{L}\{y(t)\} = Y(s)$ then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)$$

⋮

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t \, dt \overset{\text{IBP}}{=} \cdots = \frac{1}{s^2}$$

In general

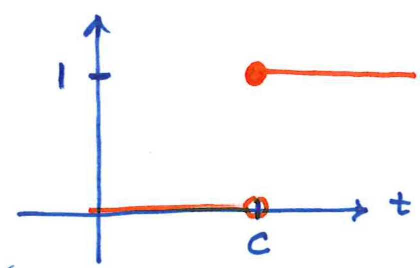
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$n \in \mathbb{N}$

One main application of LT is in solving a linear ODE with a discont's or impulsive force function.

One of the important functions with a jump dis continuity is unit step function or Heaviside function :

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



a jump of 1 unit at $t = c$

$1 \rightsquigarrow$ "on"

$0 \rightsquigarrow$ "off"

$\Rightarrow u_c(t)$: a switch that is turned on at time c.

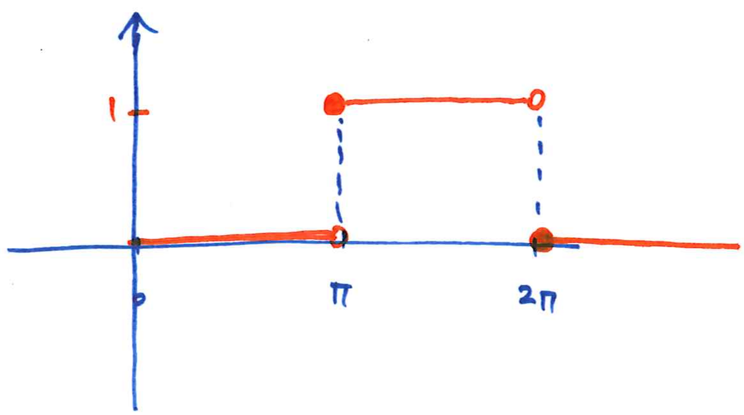
$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_c^\infty \underset{s>0}{=} \frac{e^{-cs}}{s}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

Ex 1. Sketch $h(t) = u_\pi(t) - u_{2\pi}(t)$ and find $\mathcal{L}\{h(t)\}$.

$$h(t) = \begin{cases} 0 & t < \pi \\ 1 & t \geq \pi \end{cases} - \begin{cases} 0 & t < 2\pi \\ 1 & t \geq 2\pi \end{cases}$$

$$= \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

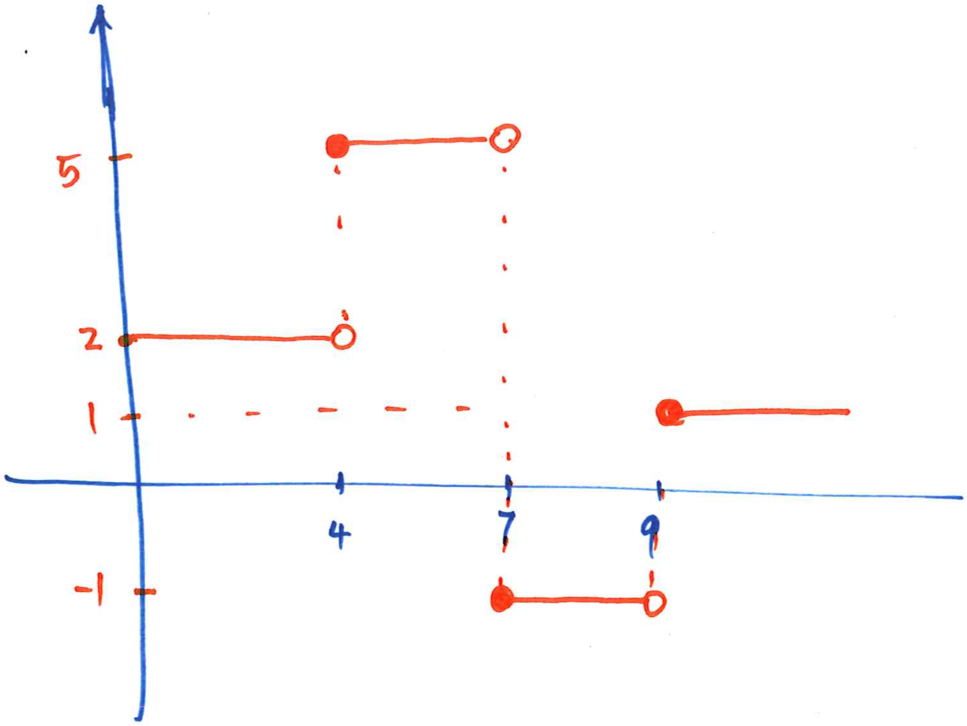


rectangular pulse

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u_\pi(t) - u_{2\pi}(t)\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

Ex 2. Sketch $f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & t \geq 9 \end{cases}$ and express

f in terms of $u_c(t)$ and find $\mathcal{L}\{f(t)\}$.



Start with $f_1(t) = 2$

then a jump of 3 units at $t = 4$

$$2 + 3\,u_4(t)$$

then a jump of -6 units at $t = 7$

$$2 + 3\,u_4(t) - 6\,u_7(t)$$

then a jump of 2 units at $t = 9$

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

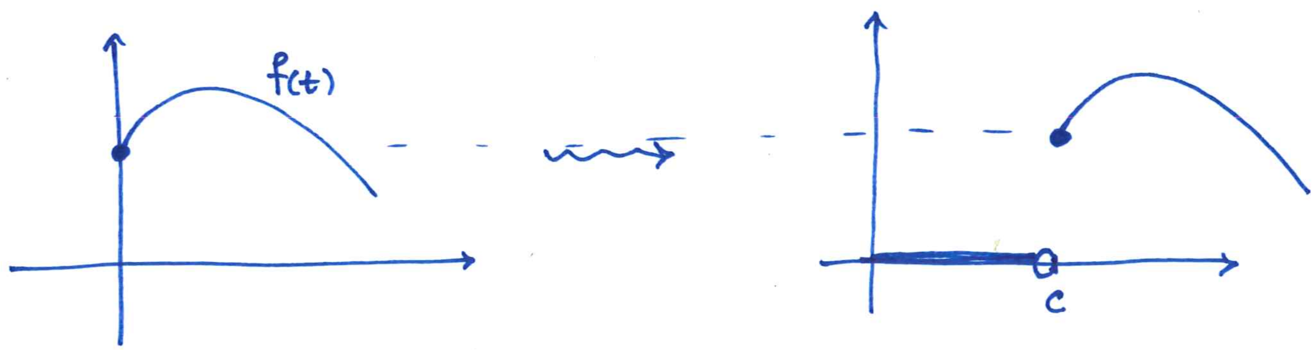
$$\mathcal{L}\{f(t)\} = \frac{2}{s} + 3\,\frac{e^{-4s}}{s} - 6\,\frac{e^{-7s}}{s} + 2\,\frac{e^{-9s}}{s}$$

Given $f(t)$; $\quad g(t) = f(t-c)$

$\swarrow c>0$

shifting f c units to the right

Now define $\quad g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$



$$\Rightarrow \quad g(t) = u_c(t)\, f(t-c)$$

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = \int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\, dt$$

$t - c = r$

$dt = dr$

$t = r + c$

$t = c \rightsquigarrow r = 0$

$$= \int_c^\infty e^{-st}\, f(t-c)\, dt$$

$$= \int_0^\infty e^{-s(r+c)} \cdot f(r)\, dr$$

$$= e^{-sc} \int_0^\infty e^{-sr}\, f(r)\, dr = e^{-sc}\, \mathcal{L}\{f\}$$

$$\boxed{\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-sc} F(s)}$$

Shift f c units $\rightsquigarrow$ multiply F by e^{-sc}

Ex 3. $f(t) = \begin{cases} \sin t & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \pi/4) & t \ge \frac{\pi}{4} \end{cases}$, find $F(s)$.

$$= \sin t + \begin{cases} 0 & 0 \le t < \frac{\pi}{4} \\ \cos(t - \frac{\pi}{4}) & t \ge \frac{\pi}{4} \end{cases}$$

$$= \sin t + u_{\pi/4}(t) \cos(t - \pi/4)$$

$f(t - \pi/4)$

$f(t) = \cos t$

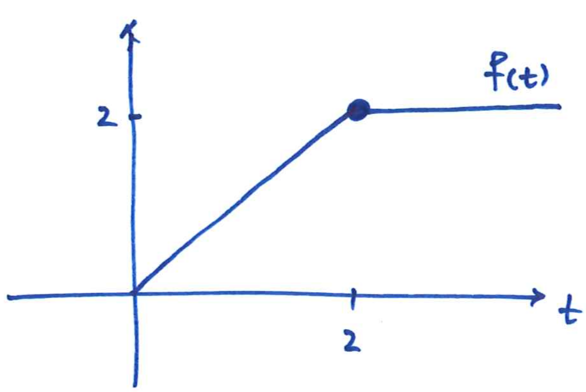
$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi/4\, s} \frac{s}{s^2+1}$$

e^{-cs} $F(s)$

Ex 4: $\mathcal{L}^{-1}\left\{ \dfrac{1 - e^{-2s}}{s^2} \right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{ e^{-2s} \frac{1}{s^2} \right\}$

$$= t - u_2(t)(t-2)$$

$f(t) = t$



$$f(t) = \begin{cases} t & 0 \le t < 2 \\ 2 & t \ge 2 \end{cases}$$

Ex5 : Solve $y'' + 4y = g(t)$

$$y(0) = 0 = y'(0)$$

where
$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1 & t \geq 10 \end{cases}.$$

First write $g(t)$ in terms of $u_c(t)$:

$$g(t) = 0 + \frac{1}{5}u_5(t)\,(t-5) + u_{10}(t)\boxed{-\frac{1}{5}(t-10)}$$

when $t \geq 10$

$$\Rightarrow g(t) = 1 \Rightarrow \frac{1}{5}(t-5) + \boxed{} = 1$$

$$\Rightarrow \boxed{} = 1 - \frac{1}{5}(t-5)$$

$$= -\frac{1}{5}(t-10)$$

Apply LT:

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4Y(s) = \frac{1}{5}e^{-5s}\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$$

$$Y(s) = \frac{1}{5}\,\frac{e^{-5s} - e^{-10s}}{s^2(s^2+4)}$$

$$= \frac{1}{5}\left(e^{-5s}\underbrace{\frac{1}{s^2(s^2+4)}}_{F(s)} - e^{-10s}\underbrace{\frac{1}{s^2(s^2+4)}}_{F(s)}\right)$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2(s^2+4)} \right\} = \mathcal{L}^{-1}\left\{ \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \right\}$$

You do! $= \mathcal{L}^{-1}\left\{ \frac{1/4}{s^2} - \frac{\frac{1}{4}\,2}{2\,(s^2+4)} \right\}$

$$f(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{5}\Big(u_5(t)\cdot f(t-5) - u_{10}(t)\, f(t-10) \Big) = y(t)$$

$$= \begin{cases} 0 & t < 5 \\[2ex] \frac{1}{5}\left(\frac{1}{4}(t-5) - \frac{1}{8}\sin(2t-10)\right) & 5 \le t < 10 \\ \quad = \frac{1}{20}t - \frac{1}{40}\sin(2t-10) - \frac{1}{4} & \\[2ex] \frac{1}{5}\left(\frac{1}{4}(t-5) - \frac{1}{8}\sin(2t-10)\right) - \frac{1}{5}\left(\frac{1}{4}(t-10) - \frac{1}{8}\sin(2t-20)\right) & \\ \quad = \frac{1}{4} - \frac{1}{40}\Big(\sin(2t-10) - \sin(2t-20)\Big) & t \ge 10 \end{cases}$$

Justify this solution with the method of undetermined coefficients.