Announcements :

- A practice WW for Laplace Transform is posted.
  ↳ NOT for grades.

- Last year's Exam will be posted.

- Check the Math Department website for past exams.

  Undergraduates tab → Final Exams → Past Exams.

- **Office Hours Next Week :**

  in MATX 1118
  - Thursday Dec 6 : 11 am – 1 pm, 3 pm – 5 pm
  - Friday Dec 7 : 10 – 11 am

  Or email me for an appointment.

- **Final Exam :**

  Friday, Dec 7 at 12:00 pm

  Duration : 2.5 hours

  Location : OSBO A

- Complete the teaching Evaluation Survey on Canvas by Dec 3.

Nov 28

Lecture 34

Dirac delta function:

Consider the following step function

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & -\varepsilon/2 < t < \varepsilon/2 \\ 0 & t \geq \varepsilon/2 \text{ or } t \leq -\varepsilon/2 \end{cases}$$

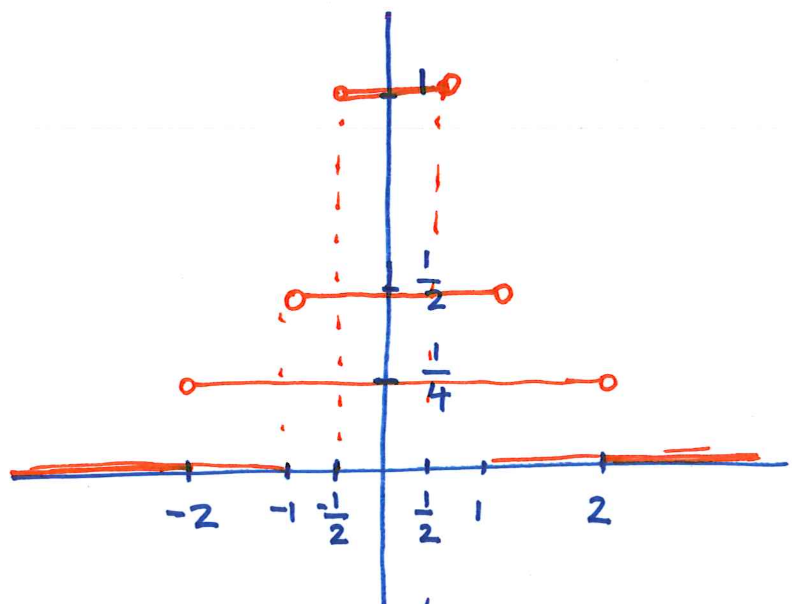
For example:

$\varepsilon = 4$

$\varepsilon = 2$

$\varepsilon = 1$

ε is a small positive number.



a step function with step width is ε and height $\frac{1}{\varepsilon}$.

We see that

$$I(\varepsilon) = \int_{-\infty}^{\infty} f_\varepsilon(t)\, dt = 1 \quad \varepsilon \neq 0$$

Now shrink the step width : $\varepsilon \to 0^+$

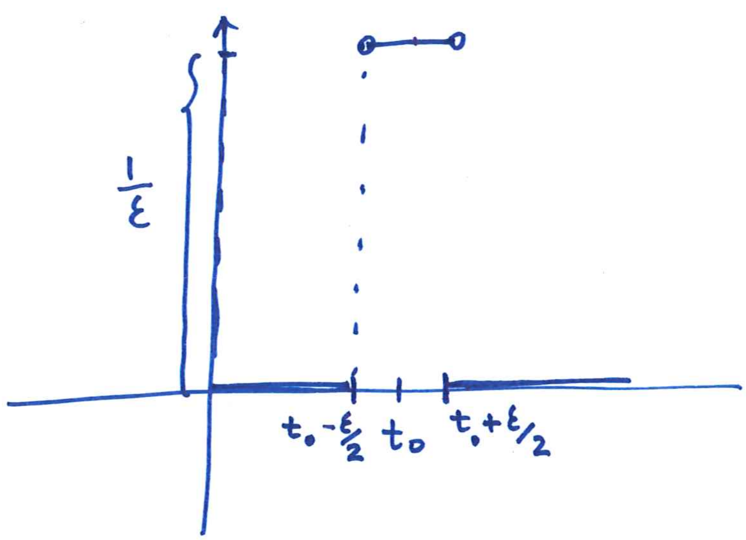
then $\lim_{\varepsilon \to 0^+} f_\varepsilon(t) = 0$

and still $\lim_{\varepsilon \to 0^+} I(\varepsilon) = 1$

The limit of a step function such as f_ε when $\varepsilon \to 0^+$ is called unit impulse function or Dirac delta function, and it is denoted by $\delta(t)$.

$$
\begin{cases}
\delta(t) = \lim\limits_{\varepsilon \to 0^+} f_\varepsilon(t) = 0 & t \neq 0 \\[2ex]
\text{and} \quad \displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1
\end{cases}
$$

In general for any t_0



$$
f_\varepsilon(t - t_0) = \begin{cases} \dfrac{1}{\varepsilon} & t_0 - \dfrac{\varepsilon}{2} < t < t_0 + \dfrac{\varepsilon}{2} \\[2ex] 0 & \text{elsewhere} \end{cases}
$$

then $\delta(t - t_0) = \lim\limits_{\varepsilon \to 0^+} f_\varepsilon(t - t_0) = 0$ when $t \neq t_0$

$$
\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1
$$

No traditional function satisfies these properties.
$\delta(t)$ or $\delta(t - t_0)$ is a generalized function or a distribution.

To find $\mathcal{L}\{\delta(t-t_0)\}$ we use its limit definition:

$$\delta(t-t_0) = \lim_{\varepsilon \to 0^+} f_\varepsilon(t-t_0) \qquad t_0 > 0$$

$$\mathcal{L}\{f_\varepsilon(t-t_0)\} = \int_0^\infty e^{-st} f_\varepsilon(t)\, dt$$

$$= \int_{t_0-\varepsilon/2}^{t_0+\varepsilon/2} \frac{1}{\varepsilon} e^{-st}\, dt$$

$$= \frac{1}{\varepsilon} \; -\frac{1}{s} e^{-st} \Bigg|_{t_0-\varepsilon/2}^{t_0+\varepsilon/2}$$

$$= \frac{1}{\varepsilon}\left(-\frac{1}{s} e^{-(t_0+\varepsilon/2)s} + \frac{1}{s} e^{-(t_0-\varepsilon/2)s} \right)$$

$$= \frac{1}{s} e^{-t_0 s} \left(\frac{1}{\varepsilon} \left(e^{\varepsilon/2\, s} - e^{-\varepsilon/2\, s}\right) \right)$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{s} e^{-t_0 s} \frac{e^{\varepsilon/2\, s} - e^{-\varepsilon/2\, s}}{\varepsilon} = \frac{0}{0}\,!$$

Apply l'Hôpital's Rule (differentiate w.r.t ε)

$$\frac{1}{s} e^{-t_0 s} \lim_{\varepsilon \to 0^+} \frac{\frac{s}{2} e^{\varepsilon s/2} + \frac{s}{2} e^{-\frac{\varepsilon}{2} s}}{1}$$

$$= e^{-t_0 s}$$

$$\Rightarrow \boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

Ex1: Solve $\quad 2y'' + y' + 2y = \delta(t-5)$

$$y(0) = 0 = y'(0)$$

Apply LT: $\quad 2s^2 Y(s) + s\,Y(s) + 2\,Y(s) = e^{-5s}$

$$\Rightarrow Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

$\hookrightarrow b^2 - 4ac = 1 - 16 = -15 < 0$

$\Rightarrow$ No factorization

$\Rightarrow$ No partial fraction

Recall: Completing the square

$$ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$\Rightarrow \quad 2s^2 + s + 2 = 2\left(s^2 + \frac{1}{2}s + 1\right) = 2\left[\left(s + \frac{1}{4}\right)^2 - \frac{1}{16} + 1\right]$$

$$Y(s) = \frac{1}{2}\,\frac{e^{-5s}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}} = \frac{1}{2}\, e^{-6s}\, F(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

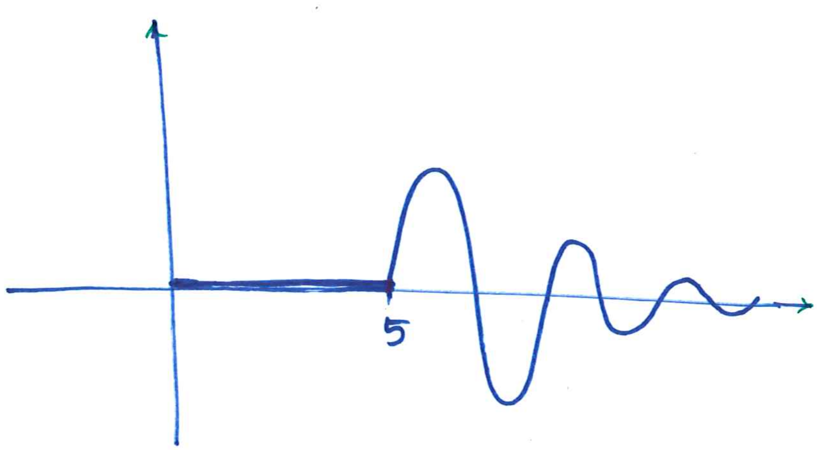
$$\frac{1}{2}\, u_5(t)\, f(t-5)$$

$$\mathcal{L}^{-1}\left\{ \frac{4}{\sqrt{15}} \cdot \frac{1 \cdot \frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}} \right\} =$$

$$\frac{4}{\sqrt{15}}\, e^{-\frac{1}{4}t} \sin\left(\frac{\sqrt{15}}{4} t\right) = f(t)$$

$$\Rightarrow \quad y(t) = \frac{2}{\sqrt{15}}\, u_5(t)\, \underbrace{e^{-\frac{1}{4}(t-5)} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)}$$

$$= \begin{cases} 0 & t < 5 \\ \frac{2}{\sqrt{15}} & t \geq 5 \end{cases}$$



Ex 2: Solve $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$

$y(0) = 0$

$y'(0) = \frac{1}{2}$

$$Y(s) = \underbrace{\boxed{\frac{e^{-5s}}{s^2+3s+2}}}_{\mathrm{I}} + \underbrace{\boxed{\frac{e^{-10s}}{s(s^2+3s+2)}}}_{\mathrm{II}} + \underbrace{\boxed{\frac{1}{2}\,\frac{1}{s^2+3s+2}}}_{\mathrm{III}}$$

$$\frac{1}{s^2+3s+2} = \frac{A}{s+2} + \frac{B}{s+1}$$

$(s+2)(s+1)$

$$= \frac{-1}{s+2} + \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{\ \} = -e^{-2t} + e^{-t}$$

$$\mathcal{L}^{-1}\{\mathrm{I}\} = u_5(t)\, f(t-5) = u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right)$$

$$\mathcal{L}^{-1}\{\mathrm{III}\} = \frac{1}{2}\left(e^{-t} - e^{-2t}\right)$$

$$\mathrm{II}: \quad \frac{1}{s(s^2+3s+2)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+2} - \frac{1}{s+1}$$

$$\xrightarrow{\mathcal{L}^{-1}} \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

$$\mathcal{L}^{-1}\left\{ e^{-10s}\frac{1}{s(s^2+3s+2)} \right\} = u_{10}(t)\, f(t-10)$$

$$= u_{10}(t)\left(\frac{1}{2}+\frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right)$$

$\Rightarrow$ Collect everything :

$$y(t) = u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right)$$
$$+ u_{10}(t)\left(\frac{1}{2}+\frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right)$$
$$+ \frac{1}{2}\left(e^{-t} - e^{-2t}\right)$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$		$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\frac{1}{s}$	(1)	e^{at}	$\frac{1}{s-a}$	(13)
$e^{at}f(t)$	$F(s-a)$	(2)	te^{at}	$\frac{1}{(s-a)^2}$	(14)
$u_a(t)$	$\frac{e^{-as}}{s}$	(3)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	(15)
$f(t-a)u_a(t)$	$e^{-as}F(s)$	(4)	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$	(16)
$\delta(t-t_0)$	e^{-st_0}	(5)	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$	(17)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	(6)	$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$	(18)
$f'(t)$	$sF(s)-f(0)$	(7)	$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$	(19)
$f^n(t)$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	(8)	$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$	(20)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(9)	$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$	(21)
$t^n \ (n=0,1,2,\dots)$	$\frac{n!}{s^{n+1}}$	(10)	$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$	(22)
$\sin kt$	$\frac{k}{s^2+k^2}$	(11)	$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$	(23)
$\cos kt$	$\frac{s}{s^2+k^2}$	(12)			