Lecture 35 (Last Class ☹)

Reminders:

- Final Exam: Friday, Dec 7, at 12:00 pm in OSBO A
- Office Hours next week: Thurs, Dec 6, 11am–1 pm, 3–5 pm; Friday, Dec 7, 10–11 am — in MATX 1118
- Exam covers all topic Week 1 – Week 13
- Make sure you know the pre-req topics:
  - → linear algebra: eigval, eigvec, generalized eigvec
    - determinant (2x2 & 3x3 matrix)
    - inverse matrix (2x2 & 3x3 matrix)
    - matrix multiplication ...
  - → Elementary algebra of complex numbers.
  - Of Course → Differentiation & Integration
- TEACHING EVALUATION → Please complete the survey ☺

MATH 215/255 Summary

ODE

1-dim (single)

Solution: Scalar 1D function $y(t)$, $x(t)$, $y(x)$, ...

- 1^{st}-order → General form: $\boxed{y'(t) = f(t, y)}$
- 2^{nd} order → General form: $\boxed{y''(t) = f(t, y, y')}$

n-dim (system)

Solution: n-dim vector with n components $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

- linear → General form: $\boxed{\vec{x}'(t) = A(t)\vec{x}(t) + \vec{G}(t)}$
- Non linear → General form: $\boxed{\vec{x}'(t) = F(t, \vec{x}(t))}$

$X(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$

1st branch
1 dim ODE
(Scalar function)

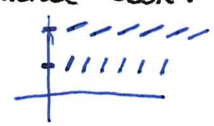
1st order

- Existence & uniqueness of sol'n ⇝ Picard
- largest interval of existence ⇝ linear / non linear

Types &
- Methods to solve analytically
  - (1) — Separable ODE ⇝ linear or nonlinear
  - (2) — Integrating factor ⇝ works for all linear eqt.
  - (3) — Exact ODE $M\,dx + N\,dy = 0$ ⇝ $M_y = N_x$ ⇝ Non-exact
  - (4) — Laplace transform

applied problems
- Falling object model
- Concentration of chemicals in water → textbook & WW

- Autonomous : $\boxed{y'(t) = f(y)}$ → Direction field general look : (slope)



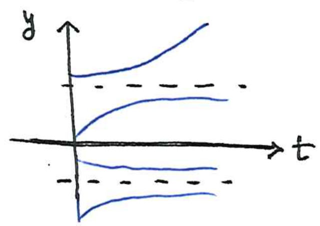
Critical points $\xrightarrow{\text{means}}$ $\boxed{y'(t) = 0 = f(y)}$ → graph of y' vs. y

↓

Phase line
- asym. stable (y_0)
- semi-stable (y_0)
- unstable (y_0)

Use this to analyze the solution behaviour.

- Solution is 1D : $y(t)$ ⇝ dep. variable can be graphed vs. the indep variable



- Numerical methods
  - Euler & Improved Euler
    - algorithm & process
    - error orders
    - MATLAB Implementation

2nd order

linear with constant coeff

$ay'' + by' + cy = g(t)$

↓

Characteristic eqt.

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $g(t) = 0$ Homogeneous
- $g(t) \neq 0$ Non-hom
  - Variation of param.
  - Undeter. Coeff.

A special case $a, b, c > 0$ ($b \geq 0$)

- unforced ($g = 0$)
  - undamp: $m x'' + kx = 0$ → natural freq $\omega_0 = \sqrt{\frac{k}{m}}$, $T = \frac{2\pi}{\omega_0}$ → pure oscillation
  - damp:
    - • underdamped → decaying oscillation
    - • overdamped → NO oscillation fast decay
    - • critically damp → NO oscillation fast decay
- forced ($g \neq 0$)
  - undamp: $m x'' + kx = f(t)$ → sin/cos → Resonance & Beats
  - damp: Full solution = transient sol'n (x_H) + force response (x_p) → dominant

The other branch

n-dim ODE systems (Vector of solutions)

linear

$\vec{x}' = A(t)\,\vec{x} + \vec{G}(t)$

Solution form: $\boxed{\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + \cdots + c_n\vec{X}_n(t)}$

Hom. autonomous

$\vec{x}' = A\vec{x} \xrightarrow{\det A \neq 0} (0,0)$: the only critical point

eigval & eigvec

- real
  - distinct
    - $r_1 \neq r_2$ nonzero
    - $r_1 \neq r_2$ one is 0
  - repeated
    - $r_1 = r_2 = r \neq 0$
    - $r_1 = r_2 = r = 0$
- Complex

→ Solution form: $\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + c_2\vec{v}_2 e^{r_2 t}$

same for higher dim

repeated eigval

- Stability of $(0,0)$
  - Asym. stable: All solutions $\xrightarrow{t\to\infty} (0,0)$
  - Stable: stay close to $(0,0)$ but NOT converge
  - Unstable: diverge from $(0,0)$
- Type of $(0,0)$
- Fundamental matrix for each case & Wronskian

  $\mathbf{X}(t) = (x_1 \,\vdots\, x_2 \,\vdots\, \cdots \,\vdots\, x_n)$
- e^{tA} $\xrightarrow{\text{two methods}}$ (1) diagonalization; (2) $= \mathbf{X}(t)\,\mathbf{X}^{-1}(0)$
- Non-homogeneous: $\vec{x}' = A\vec{x} + \vec{G}(t) \rightsquigarrow \vec{x} = \vec{x}_H + \vec{x}_p$

  Find $\vec{x}_p$

  - Variation of Parameters

    $\boxed{\vec{x}_p = \mathbf{X}(t)\left(\int \mathbf{X}^{-1}(t)\,G(t)\,dt\right)}$
  - Undeter. Coeff only works when

    $G(t)$: polynomial / constant, exponential, Sin or Cos

Non linear

$\vec{x}' = \vec{F}(t, \vec{x}(t))$

autonomous: $\vec{x}' = \vec{F}(\vec{x}(t))$

2D: $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ $\xrightarrow{\text{critical points}}$ More than one usually $f(x,y) = 0 = g(x,y)$

Find x-nullcline: $x' = 0 \to x_1, x_2, \ldots$

Plug them into g, solve for $y \to y_1, y_2, \ldots$

- linearization around each critical point:

  $J(x_0,y_0) = DF(x_0,y_0) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{(x_0,y_0)}$

  * $\det J(x_0,y_0) \neq 0$ isolated

  $\Rightarrow \boxed{\vec{x}' = DF(x_0,y_0)\,\vec{x}}$
- Conclusion about the nonlinear system:

  Table

  Theorem: Near the critical point $\vec{x}_0$ i.e. when $|x - x_0|$ is small, the nonlinear terms are also small and do not affect the stability & type of critical points as determined by the linearized system, EXCEPT in two cases:

  - linear: $r_{1,2} = \pm i\beta$ → nonlinear: $\alpha \pm i\beta'$ $(\beta \approx \beta')$

    Stability & type both change.
  - linear: $r_1 = r_2 = r$ → nonlinear: $r_1 \neq r_2$

    Only type changes, the same stability.

Practice Problems: (Laplace & Non linear systems)

1) (i) Sketch the following functions.
   (ii) Express the functions a-f in terms of the unit step function.

(a) $f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -2, & 3 \le t < 5, \\ 2, & 5 \le t < 7, \\ 1, & t \ge 7. \end{cases}$

(b) $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2, \\ 1, & 2 \le t < 3, \\ -1, & 3 \le t < 4, \\ 0, & t \ge 4. \end{cases}$

(c) $f(t) = \begin{cases} 1, & 0 \le t < 2, \\ e^{-(t-2)}, & t \ge 2. \end{cases}$

(d) $f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & 2 \le t < 5, \\ 7-t, & 5 \le t < 7, \\ 0, & t \ge 7. \end{cases}$

(e) $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$

(f) $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

(g) $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

(h) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

(i) $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$

(j) $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$

(k) $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

2) Find the inverse Laplace transform of the following functions.

$$F(s) = \frac{3!}{(s-2)^4}$$

$$F(s) = \frac{e^{-2s}}{s^2 + s - 2}$$

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$$

$$F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$$

3) Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

a. Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \quad s > ca.$$

b. Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

c. Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

4) Solve the following initial-value problems.

1. $y'' + y = f(t); \quad y(0) = 0, \; y'(0) = 1;$
$$f(t) = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & 3\pi \leq t < \infty \end{cases}$$

2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \; y'(0) = 1;$
$$h(t) = \begin{cases} 1, & \pi \leq t < 2\pi \\ 0, & 0 \leq t < \pi \quad \text{or} \quad t \geq 2\pi \end{cases}$$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \quad y(0) = 0, \; y'(0) = 0$

4. $y'' + 3y' + 2y = f(t); \quad y(0) = 0, \; y'(0) = 0;$
$$f(t) = \begin{cases} 1, & 0 \leq t < 10 \\ 0, & t \geq 10 \end{cases}$$

5. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)\left(t - \frac{\pi}{2}\right); \quad y(0) = 0, \; y'(0) = 0$

6. $y'' + y' + \frac{5}{4}y = g(t); \quad y(0) = 0, \; y'(0) = 0;$
$$g(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

7. $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t); \quad y(0) = 0, \; y'(0) = 0$

5) Solve the following initial-value problems.

1. $y'' + 2y' + 2y = \delta(t-\pi);\quad y(0) = 1,\ y'(0) = 0$
2. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi);\quad y(0) = 0,\ y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t);\quad y(0) = 0,\ y'(0) = 1/2$
4. $y'' + 2y' + 3y = \sin t + \delta(t-3\pi);\quad y(0) = 0,\ y'(0) = 0$
5. $y'' + y = \delta(t-2\pi)\cos t;\quad y(0) = 0,\ y'(0) = 1$
6. $y'' + 4y = 2\delta(t-\pi/4);\quad y(0) = 0,\ y'(0) = 0$
7. $y'' + 2y' + 2y = \cos t + \delta(t-\pi/2);\quad y(0) = 0,\ y'(0) = 0$
8. $y^{(4)} - y = \delta(t-1);\quad y(0) = 0,\ y'(0) = 0,$ $y''(0) = 0,\ y'''(0) = 0$

6) (a) For each of the following nonlinear systems find the equilibrium solutions and the linear system near each critical point.
(b) Use the linear system, if possible, to draw conclusions about the stability and type of the equilibria of the nonlinear system.

a) $dx/dt = (2+x)(y-x),\quad dy/dt = (4-x)(y+x)$

b) $dx/dt = x - x^2 - xy,\quad dy/dt = 3y - xy - 2y^2$

c) $dx/dt = 1 - y,\quad dy/dt = x^2 - y^2$

d) $dx/dt = (2+y)(2y-x),\quad dy/dt = (2-x)(2y+x)$

e) $dx/dt = x + x^2 + y^2,\quad dy/dt = y - xy$

f) $dx/dt = (1+x)\sin y,\quad dy/dt = 1 - x - \cos y$

g) $dx/dt = x - y^2,\quad dy/dt = y - x^2$

h) $dx/dt = 1 - xy,\quad dy/dt = x - y^3$

i) $dx/dt = -2x - y - x(x^2+y^2),$
$dy/dt = x - y + y(x^2+y^2)$

j) $dx/dt = y + x(1-x^2-y^2),\quad dy/dt = -x + y(1-x^2-y^2)$

k) $dx/dt = 4 - y^2,\quad dy/dt = (1.5+x)(y-x)$

l) $dx/dt = (1-y)(2x-y),\quad dy/dt = (2+x)(x-2y)$

$y_1 = \frac{1}{t}, \ y_2 = \frac{1}{t^2} \qquad t \neq 0$

$$\det \begin{pmatrix} \frac{1}{t} & \frac{1}{t^2} \\ -\frac{1}{t^2} & -\frac{2}{t^3} \end{pmatrix} = -\frac{2}{t^4} + \frac{1}{t^4} = -\frac{1}{t^4} \neq 0$$

$W[y_1, y_2] \neq 0 \iff y_1, y_2$ linearly independent

(d) $f(t) = t + (2-t)\,u_2(t) + (7-t) - 2 \quad u_5(t)$

$+ \ 0 - (7-t) \ u_7(t)$

$$* \quad f(t) = \begin{cases} f_1(t) & 0 \leq t < a \\ f_2(t) & a \leq t \leq b \\ f_3(t) & t \geq b \end{cases}$$

$$f(t) = f_1(t) + \left(f_2(t) - f_1(t)\right) u_a(t) + \left(f_3(t) - f_2(t)\right) u_b(t)$$