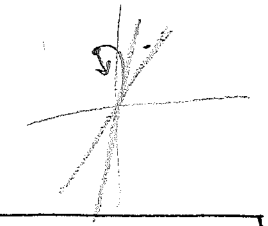
Review Notes : Midterm 2

Systems of 1^{st} order ODE : $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

1

-3



Find eigval & eigvec of A and

$$X(t) = c_1 \vec{v}_1 e^{r_1 t} + c_2 \vec{v}_2 e^{r_2 t}$$

Real case

r_1 & r_2 real, distinct

$r_1 \neq r_2$, $r_1 r_2 \neq 0$

⇓

Reviewed in MT 1.

$r_1 \neq r_2$, $r_1 = 0$

⇓

$X(t) = c_1 v_1 + c_2 v_2 e^{r_2 t}$

⇓

One piece of the sol'n does NOT depend on t.

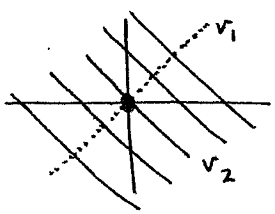
⇓

infinitely many equilibria all on the line through v_1

⇓

trajectories are straight lines parallel to v_2; each of these lines crosses an equilibrium.

*NO arrows on v_1 : all v_1 is equilibrium.



* $r_2 > 0 \rightsquigarrow$ arrows out $\rightsquigarrow (0,0)$ unstable
* $r_2 < 0 \rightsquigarrow$ arrows in $\rightsquigarrow (0,0)$ stable

repeated $r_1 = r_2 \neq 0$

only one eigvec v

⇓

need a generalized eigvec : w that solves

$(A - rI)\vec{w} = \vec{v}$

⇓

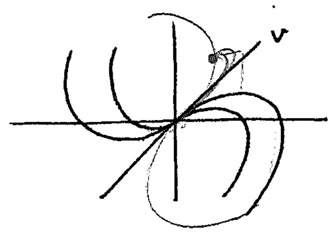
$X(t) = c_1 v e^{rt} + c_2 (tv + w) e^{rt}$

⇓

one piece of the sol'n has te^{rt} piece.

⇓

We only sketch v NOT w



* $r > 0 \rightsquigarrow$ arrows out $\rightsquigarrow (0,0)$ unstable
* $r < 0 \rightsquigarrow$ arrows in $\rightsquigarrow (0,0)$ asym. stable

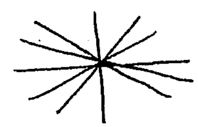
two linearly indep eigvec v_1 & v_2

⇓

$X(t) = c_1 v_1 e^{rt} + c_2 v_2 e^{rt}$

⇓

star-shaped trajectories



* $r < 0 \rightsquigarrow$ inward
* $r > 0 \rightsquigarrow$ outward

* This case occurs when $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I$ and $r = \alpha$.

complex case

$r_{1,2} = \alpha \pm i\beta$

take one eigval and find its eigvec and write:

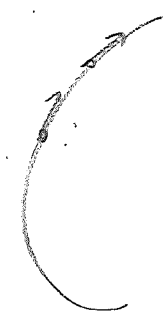
$$X_1(t) = v_1 e^{\alpha t} \cdot e^{i\beta t}$$
$$= v_1 e^{\alpha t} (\cos\beta t + i \sin\beta t)$$

distribute v_1, seperate real and imaginary parts of x_1 so that

$$X_1(t) = U(t) + i V(t)$$

then the general sol'n is

$$X(t) = c_1 U(t) + c_2 V(t)$$



Fundamental Matrix :

Find two linearly independent solutions x_1 and x_2 and make a matrix whose columns are x_1 and x_2 : $\Phi(t)$ or $\mathbb{X}(t) = \left(x_1(t) \,\vdots\, x_2(t) \right)$

→ In each of the cases in the previous diagram, each piece of the general solution (without constants c_1 and c_2) is a column in the fundamental matrix.

Matrix Exponential : e^{tA}

two definition for e^{tA}

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

⇓

works when A is diagonal or diagonalizable

↓

check lecture notes for examples.

$$e^{tA} = \mathbb{X}(t)\,\mathbb{X}^{-1}(0)$$

⇓

This method works for any matrix A.

↓

especially when A is NOT diagonalizable which is when A has repeated eigval with only one eigvec.

↓

Check the example in the lecture notes

Non-homogeneous Systems

$$\vec{x}' = A\vec{x} + G(t) \Rightarrow \text{Solution: } X(t) = X_H(t) + X_p(t)$$

$X_H(t)$: Homog. sol'n ← we know how to find this :

$X_p(t)$: a solution that solves the non-hom eqt.

Finding $X_p(t)$:

Variation of Parameters : Always works

Find $\mathbb{X}(t)$ then

$$X_p(t) = \mathbb{X}(t) \int \mathbb{X}^{-1}(t)\, G(t)\, dt$$

⇓

More systematic comparing to →

Undetermined coefficients :

if $G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$ involves polynomial, exp and sin or cos

guess a general form for $X_p(t)$ based on the combinations of g_1 and g_2's

plug your guess into the system & find the coefficients.

⇓

might involve less computations comparing to ←

2

$\boxed{2^{nd} \text{ order ODE}}$

General form : $ay'' + by' + cy = 0$ (or $\div$ by a : $y'' + by' + cy = 0$)

→ Homogeneous

$ay'' + by' + cy = g(t)$

→ non-homogeneous or forced eqt.

Find the characteristic eqt :

$$ar^2 + br + c = 0$$

$$\Rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$r_{1,2}$ both real and distinct

$\Rightarrow y_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1 = r_2 = r$

$\Rightarrow y_H(t) = c_1 e^{rt} + c_2 t e^{rt}$

$r_{1,2} = \alpha \pm i\beta$

$\Rightarrow y_H(t) = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t$

* 2^{nd} order ODE is a special case of 1^{st} order 2×2 system.

$$y'' + by' + cy = g(t) \rightsquigarrow \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Compare this with $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$: Here we solve for $\begin{pmatrix} y \\ y' \end{pmatrix}$ so in the end we only pick the 1^{st} element of the solution vector, i.e. $y(t)$.

* This means the fundamental matrix $X(t)$ corresponding to the system $\begin{pmatrix} y' \\ y'' \end{pmatrix} = A \begin{pmatrix} y \\ y' \end{pmatrix}$ contains y in the 1^{st} row and y' in the 2^{nd} row.

Non-hom (forced) case : $ay'' + by' + cy = g(t)$

Find y_p :

If $g(t)$ is exp, polynomial Sin & Cos method of undetermined coefficients is usually an easier option.

Guess y_p based on $g(t)$ and roots of the characteristic equation.

For any general $g(t)$ we can use the method of variation of parameters. In this case, you need to convert the ODE to a system first, find $X(t)$ & use the formula

$$y(t) = y_H(t) + y_p(t)$$

$$\boxed{ay'' + by' + cy = g(t)}$$

known $g(t)$	guess y_p and find the coeff.
$\rightarrow P_n(t) = a_nt^n + \dots + a_1t + a_0$	(1) $A_nt^n + \dots + A_1t + A_0$: Find $A_0, A_1, \dots, A_n$
$\rightarrow ae^{\alpha t}$	(2) $Ae^{\alpha t}$
$\rightarrow P_n(t)\, e^{\alpha t}$	(3) $(A_nt^n + \dots + A_1t + A_0)\, e^{\alpha t}$
$\rightarrow e^{\alpha t}\cos\beta t$ <u>or</u> $e^{\alpha t}\sin\beta t$	(4) $Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$
$\rightarrow \cos\beta t$ <u>or</u> $\sin\beta t$	(5) $A\cos\beta t + B\sin\beta t$
$\rightarrow P_n(t)\, e^{\alpha t}\cos\beta t$ or $P_n(t)\, e^{\alpha t}\sin\beta t$	(6) $(A_nt^n + \dots + A_1t + A_0)e^{\alpha t}\cos\beta t + (B_nt^n + \dots + B_1t + B_0)e^{\alpha t}\sin\beta t$
Special Cases: $P_n(t)$ and 0 is a root of char. eqt.	$(A_nt^n + \dots + A_1t + A_0)\,t$
$e^{\alpha t}$ and α is a root of char	$Ate^{\alpha t}$
$P_n(t)\, e^{\alpha t}$ and α " " " " "	multiply (3) by t
$e^{\alpha t}\cos\beta t$ or $e^{\alpha t}\sin\beta t$ and $\alpha \pm i\beta$ are roots of char.	$(4)\cdot t$
$\cos\beta t$ or $\sin\beta t$ and $\pm i\beta$ roots of char	$(5)\cdot t$

* If $g(t) = e^{\alpha t}$ and α is a double root of char eqt we multiply our guess by t^2.

* When we had the guess for y_p; we find y_p' and y_p'', plug them into LHS of the equation, simplify and equate to RHS, compare and equate the coefficients of the corresponding terms and find $A_0, A_1, B_1, \dots$

* If you have initial conditions given $y(0)$ and $y'(0)$; you apply them at the very final step after you found y_p and formed $y(t) = y_H(t) + y_p(t)$

* If $g(t) = g_1 + g_2 + \dots + g_n \rightsquigarrow$ Find a y_p for each g_i and add them.

Mass-spring system $mx'' + cx' + kx = 0$ or $g(t)$

$m, k > 0$

$c \geq 0$

unforced

undamped

$$mx'' + kx = 0$$

$$\Downarrow$$

$$r = \pm\sqrt{\frac{k}{m}}\, i = \pm \underbrace{\omega_0}_{\text{natural freq}} i$$

Sol'n:

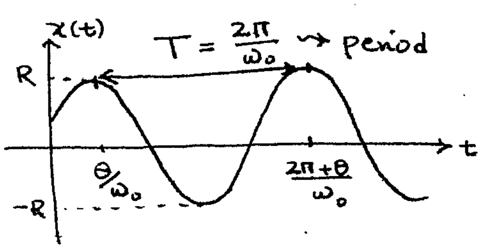
$$x_H(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

$$= R\cos(\omega_0 t - \theta)$$

$R = \sqrt{c_1^2 + c_2^2} \rightsquigarrow$ constant ampl.

$\theta = \tan^{-1}(c_1/c_2) \rightsquigarrow$ phase shift

Graph: pure indefinite oscillation



damped

$$mx'' + cx' + kx = 0$$

$$r_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

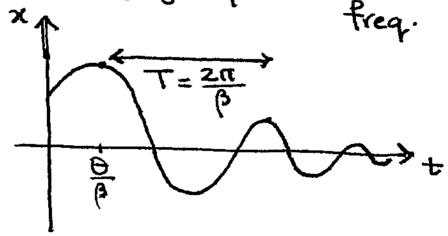
- $c^2 - 4km < 0 \rightsquigarrow$ under-damped

$\Rightarrow r_{1,2} = -\frac{c}{2m} \pm i\frac{\sqrt{4km - c^2}}{2m}$

$= -\alpha \pm i\beta$

$\Rightarrow x_H(t) = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t$

$= \underbrace{Re^{-\alpha t}}_{\text{decaying ampl.}}\cos(\underbrace{\beta t}_{\text{constant freq.}} - \theta)$



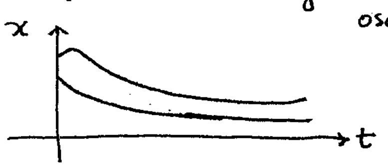
Graph: Decaying Oscillation

- $c^2 - 4km > 0 \rightsquigarrow$ over-damped

$\Rightarrow r_{1,2}$ both negative

$\Rightarrow x_H(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$\Rightarrow$ Graph: fast decay $\rightsquigarrow$ NO oscillation



- Critical damping $\rightsquigarrow c^2 - 4km = 0$

$\Rightarrow r_1 = r_2 = \frac{-c}{2m}$

$\Rightarrow x_H(t) = c_1 e^{rt} + c_2 t e^{rt}$

$\Rightarrow$ Fast decay (faster than over-damped)

forced

undamped

$$mx'' + kx = F_0\cos(\omega t) \text{ or } F_0\sin(\omega t)$$

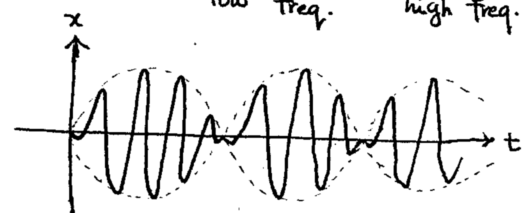
$x(t) = x_H(t) + x_p(t)$

$x_H(t)$: pure oscillation

$x_p(t)$: $\begin{cases}\omega \neq \omega_0 \\ \omega = \omega_0\end{cases}$

* $\omega \neq \omega_0$ but $\omega \approx \omega_0 \Rightarrow$ Beats

$$x(t) = R\,\underbrace{\sin\left(\frac{\omega - \omega_0}{2}\right)}_{\text{low freq.}}\,\underbrace{\sin\left(\frac{\omega + \omega_0}{2}\right)}_{\text{high freq.}}$$

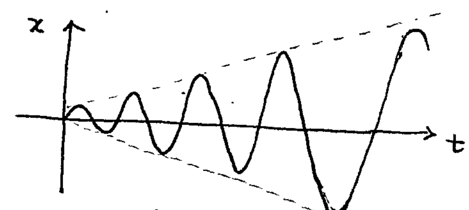


periodic ampl.

* $\omega = \omega_0 \Rightarrow$ Resonance

$x(t) = \underbrace{Rt}_{\text{growing ampl.}}\sin(\omega_0 t)$

$t \to \infty$: $x(t) \to \infty$



* c increases $\rightsquigarrow \omega_{max}$ moves to the left

$c_I < c_{II} < c_{III} < c_{IV} < c_V$

damped

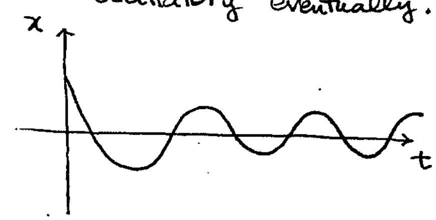
$$mx'' + cx' + kx = F_0\cos\omega \text{ or } F_0\sin\omega$$

$x(t) = x_H(t) + x_p(t)$

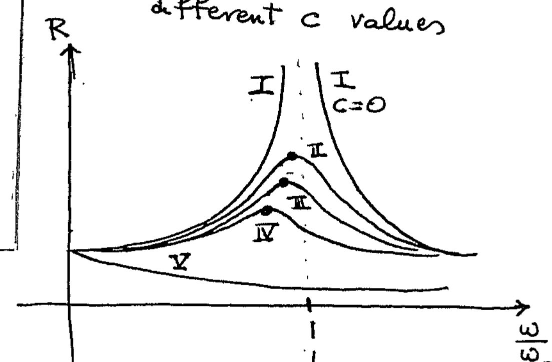
$x_H(t)$: A decaying solution $\Rightarrow$ fades away as $t \to \infty$

$\Rightarrow x_p$ becomes dominant as $t \to \infty$

* Since x_p has oscillatory nature, so solution become oscillatory eventually.



Another important graph: amplitude vs. ω for different c values



$\omega = \omega_0$ & $c = 0 \Rightarrow R$ blows up

A.

(a) Find the general solution for 1-3 and a particular solution for 4-6.
(b) Determine the stability of the origin.
(c) Find a fundamental matrix for 1-3.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

6. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

B.

For each of the following systems:
(a) Find the general solution.
(b) Find a fundamental matrix for the system.
(c) Determine the stability of the origin (stable, asymptotically stable or unstable)
(d) Sketch the phase portrait of the system.

$$x' = x - 2y, \quad y' = 3x - 6y.$$

$$x' = -x + 2y, \quad y' = -3x + 6y.$$

$$x' = -2x - 4y, \quad y' = x + 2y.$$

C.

In each of Problems 1 through 4 find the general solution of the given system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0$

D. In each of Problems 1 through 10, find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - y' - 2y = -2t + 4t^2$
3. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$
4. $y'' - 2y' - 3y = -3te^{-t}$
5. $y'' + 2y' = 3 + 4\sin(2t)$
6. $y'' + 2y' + y = 2e^{-t}$
7. $y'' + y = 3\sin(2t) + t\cos(2t)$
8. $u'' + \omega_0^2 u = \cos(\omega t), \quad \omega^2 \neq \omega_0^2$
9. $u'' + \omega_0^2 u = \cos(\omega_0 t)$
10. $y'' + y' + 4y = 2\sinh t \qquad$ *Hint:* $\sinh t = (e^t - e^{-t})/2$

E. In each of Problems 11 through 15, find the solution of the given initial value problem.

11. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
12. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
13. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
14. $y'' + 4y = 3\sin(2t), \quad y(0) = 2, \quad y'(0) = -1$
15. $y'' + 2y' + 5y = 4e^{-t}\cos(2t), \quad y(0) = 1, \quad y'(0) = 0$

F. In each of the following forced equations determine a suitable form for the particular solution that solves the equation.

$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t)$

$y'' - 5y' + 6y = e^t\cos(2t) + e^{2t}(3t + 4)\sin t$

$y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$

$y'' + 4y = t^2\sin(2t) + (6t + 7)\cos(2t)$

$y'' + 3y' + 2y = e^t(t^2 + 1)\sin(2t) + 3e^{-t}\cos t + 4e^t$

$y'' + 2y' + 5y = 3te^{-t}\cos(2t) - 2te^{-2t}\cos t$

1 • Does the solution to this equation show oscillations?

underdamping $3\frac{d^2y}{dt^2}+4\frac{dy}{dt}+y=0$ → two distinct real roots

over damped

(a) Yes.

$\boxed{\text{(b) No.}}$

2 • The motion of a mass on a spring follows the equation $M x'' + K x = 0$
where the displacement of the mass is given by x(t)
Which of the following would result in the highest frequency motion?

A. K=6, M=2 $\omega_0 = \sqrt{\frac{K}{m}}$
B. K=4, M=4 $\omega_0=\sqrt{3}$
C. K=2, M=6 $\omega_0=1$
D. K=8, M=6
E. All have the same frequency

3 • An oscillator is modeled by mx''+bx'+kx = 0. If we increase the parameter b slightly, what happens to the period of oscillation?

A. The period gets larger
B. The period gets smaller
C. The period stays the same

$\sqrt{-5} = \sqrt{-1}\cdot\sqrt{5} = i\sqrt{5}$

$\beta = \frac{\sqrt{4km-b^2}}{2m}$ increase b

β decreases → $T=\frac{2\pi}{\beta}$ increases.

4 • An oscillator is modeled by mx''+bx'+kx = 0. If we increase the parameter m slightly, what happens to the period of oscillation?

A. The period gets larger
B. The period gets smaller
C. The period stays the same

$\downarrow\beta = \frac{\sqrt{4Km-b^2}}{2m\uparrow}$ increase m

β decreases

$T = \frac{2\pi}{}$ increase

5 • An oscillator is modeled by mx''+bx'+kx = 0. If we increase the parameter k slightly, what happens to the period of oscillation?

A. The period gets larger
B. The period gets smaller
C. The period stays the same

$\beta = \frac{\sqrt{4km-b^2}}{2m}$ increase K → $\beta\uparrow$

$T\downarrow$

6. Which of the following equations has solutic exhibiting *resonance*:

A. $y'' + 2y = 10\cos(2t)$ → $\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{2}$
B. $y'' + 4y = 8\cos(2t)$ → $\omega_0 = \sqrt{4} = 2$
C. $y'' + 2y = 6\cos(4t)$ → $\omega_0 = \sqrt{2}$
D. All of the above
E. None of the above

7. Solutions of $y''+100y = 2\cos(wt)$ display *resonance* when:

$\omega_0 = \sqrt{100} = 10$

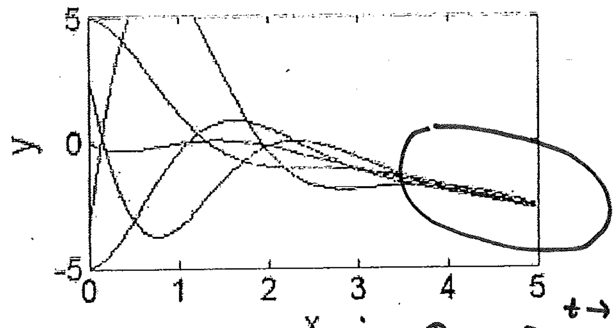
A. w = 10,000
B. w = 10
C. w = 9
D. All of the above
E. None of the above

G. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t),$$

$\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

H. The functions plotted below are solutions to which of the following differential equations?



$r_{1,2} = \alpha \pm i\beta$

$\frac{-2 \pm \sqrt{4-16}}{2}$

$t \to \infty$

negative $\leftarrow$ $Ax + B \longrightarrow -\infty$

✓(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 3 - 3x$

~~(b)~~ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 3e^{2x}$

~~(c)~~ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \sin \frac{2\pi}{9}x$

~~(d)~~ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3x - 4$

(e) None of the above

$y = \cancel{y_H} + y_p$

0

$y_p = Ae^{2x}$

I will get a positive A

$y_p = Ax^2 + Bx + C$ $\quad y_p \to +\infty$ as $t \to \infty$

positive A

$t \to \infty : \quad y_p \to \infty$