Aaron Palmer
Section 101

Nonhomogeneous Linear Second Order DE's

$$y'' + by' + cy = g(t)$$

forcing term

Two methods to find a particular solution

$c > 0$
$b \geq 0$

Undetermined Coefficients
and
Variation of Parameters

Example: $g(t) = Fe^{at}$

Find the form of the particular Soln

$\cancel{x}\, y_p = Ae^{at}$ same exponent as g.

Plug y_p into LHS of equation

$y_p' = aAe^{at}$, $y_p'' = a^2Ae^{at}$

$$a^2Ae^{at} + baAe^{at} + cAe^{at} = Fe^{at}$$

Solve for A

$$A(a^2 + ba + c) = F$$

$$A = \frac{F}{a^2 + ba + c} \quad \text{if } a^2 + ba + c \neq 0$$

for ex. $A = \dfrac{F}{a^2 + \text{[illegible]}} = \dfrac{1}{2}$

$y'' + by' + c = Fe^{at}$

Case when $a^2 + ba + c = 0$ ↳ characteristic polynomial

a is an eigenvalue.

e^{at} is a homogeneous solution.

$$(e^{at})'' + b(e^{at})' + ce^{at} = 0$$

Example

$$y'' + 3y' + 2y = e^{-2t}$$

$y(0) = 1$

$y'(0) = -1$

Characteristic polynomia

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

-2 is an eigenvalue!

Form of the particular solution is

$y_p = tAe^{-2t}$

Plug into LHS

$y_p' = -2tAe^{-2t} + Ae^{-2t}$

$y_p'' = 4tAe^{-2t} - 2Ae^{-2t} + -2Ae^{-2t}$

$= 4tAe^{-2t} - 4Ae^{-2t}$

$$y_p'' + 3y_p' + 2y_p = e^{-2t}$$

$$4tAe^{-2t} - 4Ae^{-2t} + 3(-2tAe^{-2t} + Ae^{-2t})$$

$$+ 2tAe^{-2t} = e^{-2t}$$

$$t\left[(Ae^{-2t})'' + b(Ae^{-2t})' + cAe^{-2t}\right]$$

$$(4tAe^{-2t} - 6tAe^{-2t} + 2tAe^{-2t}) \to 0$$

$$-4Ae^{-2t} + 3Ae^{-2t} = e^{-2t}$$

$$-A = 1$$

Solution is $y = -te^{-2t} + C_1 e^{-2t} + C_2 e^{-t}$

Solve for C_1, C_2 to satisfy $y(0) = 1$

$y'(0) = -1$

For an IVP

$y'' + by' + cy = Fe^{at}$ $\quad b=0$

$y(0) = 1$ $\quad c=1$

$y'(0) = 0$ $\quad F=1, \quad a=-1$

The solution is $y_p + y_H$ homogenous solution

$y'' + y = 0$

Characteristic equation (plug in $e^{\lambda t}$)

$\lambda^2 + 1 = 0, \quad \lambda = \pm i$

$$e^{it} = \underbrace{\cos(t)}_{} + i\underbrace{\sin(t)}_{}$$

two independent solutions

So $y_h = C_1 \cos(t) + C_2 \sin(t)$

Find C_1, C_2 to solve IVP

$$y = \underbrace{\tfrac{1}{2}e^{-t}}_{y_p} + C_1 \cos(t) + C_2 \sin(t)$$

$y(0) = \frac{1}{2} + C_1 + 0 = 1$

$y'(0) = -\frac{1}{2} + 0 + C_2 = 0$

$C_1 = 1/2, \quad C_2 = 1/2$

$y = \frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$ ✓

Comparison with Variation of Parameters

$$\vec{x}_p = X(t)\int X^{-1}(t)\begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix} dt$$

$$\begin{bmatrix} ae^{-2t} & be^{-t} \\ ce^{-2t} & de^{-t} \end{bmatrix} \sim \begin{bmatrix} \cancel{e^{3t}} & e^{t} \\ e^{t} & e^{t} \\ e^{2t} & e^{2t} \end{bmatrix}$$

like e^{-2t} like e^{2t} e^{-2t}

const $e^{-2t}\int dt = te^{-2t}$

If $g = e^{at}$ $a \neq 2$

Then $e^{-2t}\int e^{2t}e^{at} = e^{-2t}\,\text{cons}\, e^{2t}e^{at} = \text{const}\, e^{at}$

VoP gives the same form by integrating, but usually more work to compute the constants

General form of Particular Soln

RHS	Particular Solution
polynomial e.g t^3	polynomial $At^3 + Bt^2 + Ct + D$
exponential Fe^{at}	exponential Ae^{at} unless a is an eigenvalue then Ate^{at}.
trig e.g. $\cos(\omega t)$	trig $A\cos(\omega t) + B\sin(\omega t)$

unless ωi is an eigenvalue

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