2-a. Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}.$$

2-c. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2e^t \cos t.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = 2\,e^{(t-2)}\cos(t-2)\,u_2(t);$$

2d. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}.$$

It follows from the *translation property* of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t).$$

(3) (a). By definition of the Laplace transform,

$$\mathcal{L}[f(ct)] = \int_0^{\infty} e^{-st} f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\begin{aligned}\mathcal{L}[f(ct)] &= \frac{1}{c}\int_0^{\infty} e^{-s(\tau/c)} f(\tau)d\tau \\ &= \frac{1}{c}\int_0^{\infty} e^{-(s/c)\tau} f(\tau)d\tau\,.\end{aligned}$$

Hence $\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right)$, where $s/c > a$.

(b). Using the result in Part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f(t)]_{s\to s-c}$,

$$\mathcal{L}^{-1}[F(as+b)] = e^{-bt/a}\,\frac{1}{a} f\left(\frac{t}{a}\right).$$

4-2. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2Y(s) + 2s\,Y(s) + 2\,Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as $h(t) = u_\pi(t) - u_{2\pi}(t)$. Its transform is

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}.$$

First note that

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

Using partial fractions,

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{(s+1)+1}{(s+1)^2+1}.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}sin\,t.$$

Now let

$$G(s) = \frac{1}{s(s^2+2s+2)}.$$

Then

$$\mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2}e^{-t}cos\,t - \frac{1}{2}e^{-t}sin\,t.$$

Using ~~[illegible]~~,

$$\mathcal{L}^{-1}[e^{-cs}G(s)] = \frac{1}{2}u_c(t) - \frac{1}{2}e^{-(t-c)}[cos(t-c) + sin(t-c)]u_c(t).$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= e^{-t}sin\,t + \frac{1}{2}u_\pi(t) - \frac{1}{2}e^{-(t-\pi)}[cos(t-\pi) + sin(t-\pi)]u_\pi(t) - \\ &- \frac{1}{2}u_{2\pi}(t) + \frac{1}{2}e^{-(t-2\pi)}[cos(t-2\pi) + sin(t-2\pi)]u_{2\pi}(t). \end{aligned}$$

* This solution is for the forcing function $\sin t + u_{\pi}(t) \sin(t-\pi)$

4-3. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \mathcal{L}[h(t)].$$

$$s^2 Y(s) + 4\,Y(s) = \mathcal{L}[h(t)].$$

The transform of the forcing function is

$$\mathcal{L}[h(t)] = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t\right].$$

Based on Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[\sin(t-\pi) - \frac{1}{2}\sin(2t-2\pi)\right]u_{\pi}(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t\right] - \frac{1}{3}\left[\sin t + \frac{1}{2}\sin 2t\right]u_{\pi}(t).$$

4-4. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Hence

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}.$$

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left(1 + e^{-2(t-10)} - 2e^{-(t-10)}\right)u_{10}(t)$$

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right]u_{10}(t).$$

4-7 Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\, y(0) - y'(0) + 4\, Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\, Y(s) + 4\, Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1 ,

$$\begin{aligned} y(t) &= \frac{1}{4}[1 - cos(2t - 2\pi)]u_\pi(t) - \frac{1}{4}[1 - cos(2t - 6\pi)]u_{3\pi}(t) \\ &= \frac{1}{4}[u_\pi(t) - u_{3\pi}(t)] - \frac{1}{4} cos\, 2t \cdot [u_\pi(t) - u_{3\pi}(t)]. \end{aligned}$$

5-2. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions,

$$s^2\, Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{2} sin(2t - 2\pi)u_\pi(t) - \frac{1}{2} sin(2t - 4\pi)u_{2\pi}(t) \\ &= \frac{1}{2} sin(2t)[\, u_\pi(t) - u_{2\pi}(t)\,]. \end{aligned}$$

5-7. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + 2s\,Y(s) + 2\,Y(s) = \frac{s}{s^2+1} + e^{-(\pi/2)s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}.$$

Using partial fractions,

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right].$$

We can also write

$$\frac{s+4}{s^2+2s+2} = \frac{(s+1)+3}{(s+1)^2+1}.$$

Let

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+2)}.$$

Then

$$\mathcal{L}^{-1}[Y_1(s)] = \frac{1}{5}\,cos\,t + \frac{2}{5}\,sin\,t - \frac{1}{5}\,e^{-t}[cos\,t + 3\,sin\,t].$$

$$\Rightarrow \quad y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\left(\cos t + 3\sin t\right)$$

$$+ \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}\right\}}_{} \rightarrow \; = e^{-(t-\frac{\pi}{2})}\sin\left(t-\frac{\pi}{2}\right)u_{\frac{\pi}{2}}(t)$$

5-6. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4Y(s) = 2\,e^{-(\pi/4)s}.$$

Applying the initial conditions, $\Rightarrow \quad Y(s) = \frac{2}{s^2+4}\,e^{-(\pi/4)s}$

$$\Rightarrow \quad y(t) = sin\left(2t - \frac{\pi}{2}\right)u_{\pi/4}(t) = -\,cos(2t)\,u_{\pi/4}(t).$$