I

IT'S THE END OF THE WEEK

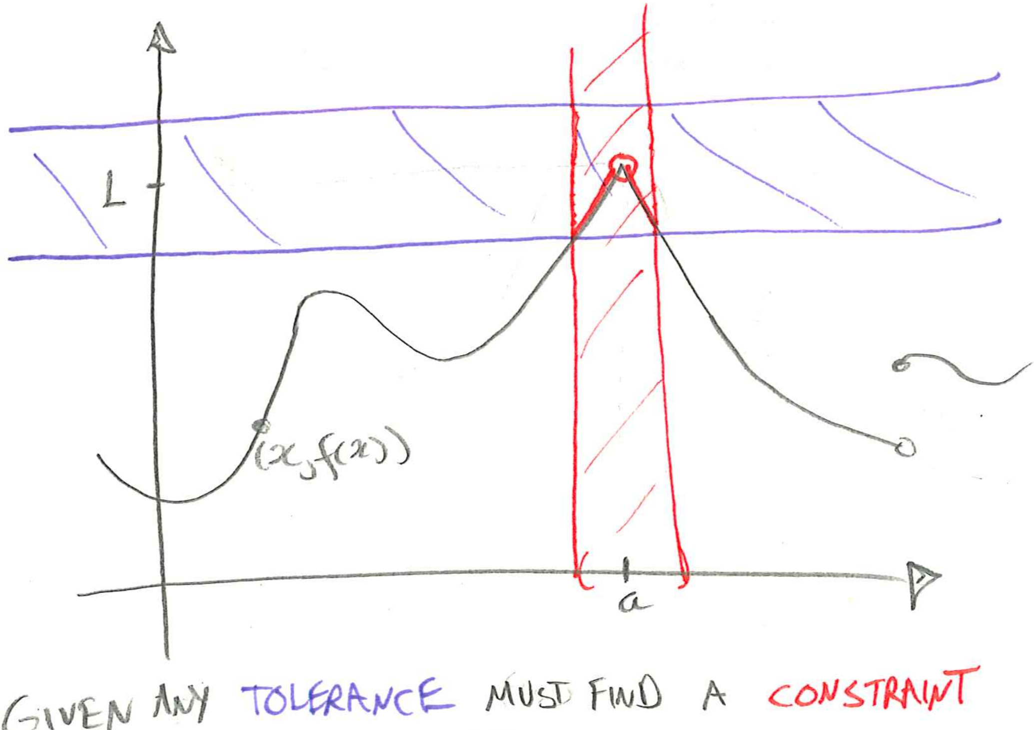
MAKE SURE YOU KNOW EVERYTHING YOU'RE SUPPOSED TO KNOW BY READING WEEK 1 LEARNING GOALS! & DOING ASSIGNED PROBLEMS.

LAST CLASS WAS ABOUT IDEAS, TODAY WE COMPUTE

"THE FUNCTION f APPROACHES THE LIMIT L NEAR A POINT a" $\lim_{x \to a} f(x) = L$

IF WE CAN MAKE $f(x)$ AS CLOSE AS WE LIKE TO L BY REQUIRING x TO BE SUFFICIENTLY CLOSE, BUT UNEQUAL, TO a.

THIS NEEDS TO BE MEMORIZED.



GIVEN ANY TOLERANCE MUST FIND A CONSTRAINT

Q: WHAT ARE LEFT / RIGHT SIDED LIMITS?

WARM UP

II

$$S(x) = \begin{cases} 1 & \text{IF } x > 0 \\ 0 & \text{IF } x = 0 \\ -1 & \text{IF } x < 0 \end{cases}$$

Q: • IS THIS A FUNCTION?

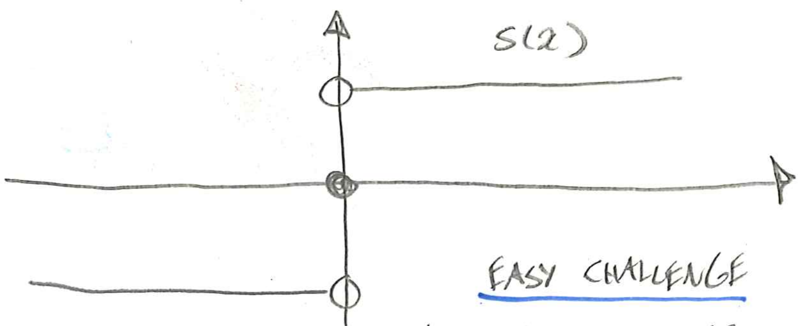
• WHAT IS $\lim\limits_{x \to 13} S(x)$?

• WHAT IS $\lim\limits_{x \to 0} S(x)$?

• WHAT IS $\lim\limits_{x \to 0^+} S(x)$?

• WHAT IS $\lim\limits_{x \to 0^-} S(x)$?

PLUG POINTS SNEAR



EASY CHALLENGE

WHAT FUNCTION IS

$$\boxed{\begin{array}{l} f(x) = S(x) \cdot |x| \\ \hline g(x) = S(x) \cdot x \end{array}} \ ?$$

THM

$$\lim_{x \to a} f(x) = L \quad \text{IF AND ONLY IF} \quad \lim_{x \to a^+} f(x) = L \ \& \ \lim_{x \to a^-} f(x) = L$$

COMPUTING A LIMIT IS HARD III

WE ARE LAZY ... SO ONCE WE KNOW ONE TRY TO GET MANY MORE OUT OF IT!

LAST TIME: TWO BASIC LIMITS

$\lim_{x \to a} c = c$	$\lim_{x \to a} x = a$

LET'S TAKE THESE GUYS FOR A RIDE:

- WHAT IS $\lim_{x \to a} (c \cdot x) =$
- WHAT IS $\lim_{x \to a} (c + x) =$

CORNER

THM IF $\lim_{x \to a} f(x) = \ell$ & $\lim_{x \to a} g(x) = m$

THEN $\lim_{x \to a} [f(x) + g(x)] = \ell + m$ (1)

& $\lim_{x \to a} [f(x) \cdot g(x)] = \ell \cdot m$ (2)

LET'S THINK ABOUT THE SUM IN MORE DETAIL

IV

- GIVEN A SMALL TOLERANCE, SAY T, NEED TO FIND CONSTRAINT ON $|x-a|$ TO MAKE $|[f(x)+g(x)]-(\ell+m)| < T$ — GOAL

- SINCE $\lim\limits_{x\to a} f(x)=\ell$ WE CAN FIND A CONSTRAINT ON $|x-a| < C_1$ TO MAKE $|f(x)-\ell| < \frac{T}{2}$

  KNOW

- SINCE $\lim\limits_{x\to a} g(x)=m$ WE CAN FIND A CONSTRAINT ON $|x-a| < C_2$ TO MAKE $|g(x)-m| < T/2$

WHAT WE KNOW

IF WE THEN LET $C < C_1$ & $C < C_2$ THEN BY REQUIRING THAT $\to C = \text{MIN}\{C_1, C_2\}$

$$|x-a| < C$$

WE ENSURE

$$|[f(x)+g(x)]-(\ell+m)| = |(f(x)-\ell)+(g(x)-m)|$$

$$\leq \underbrace{|f(x)-\ell|}_{<\frac{T}{2}} + \underbrace{|g(x)-m|}_{<\frac{T}{2}} < \frac{T}{2}+\frac{T}{2} = T$$

ENOUGH COMPLICATED STUFF, LET'S USE THM

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} x = x$$

TOOL BOX

V

WHAT IS ?

$f(x)$ $g(x)$

① $\lim_{x \to a} x^2 = \lim_{x \to a} x \cdot x \overset{①}{=} a \cdot a = a^2$

② $\lim_{x \to a} x + x \overset{②}{=} a + a$

③ $\lim_{x \to a} 3x^2 + x \overset{①}{=} \lim_{x \to a} 3x^2 + \lim_{x \to a} x$

$\overset{②}{=} 3 \lim_{x \to a} x^2 + \lim x = 3a^2 + a$

④ $\lim_{x \to a} x^3 = \lim_{x \to a} x \cdot \lim_{x \to a} x^2$

$= a \cdot a^2 = a^3$

⑤ $\lim_{x \to a} c_2 x^2 + c_1 x^1 + c_0$

$= \lim_{x \to a} c_2 x^2 + \lim_{x \to a} c_1 x^1 + \lim_{x \to a} c_0$

$= c_2 a^2 + c_1 a + c_0$

⑥ $\lim_{x \to a} c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x^1 + c_0$

$= c_n a^n + \cdots + c_1 a + c_0$

ALL POLYNOMIAL FUNCTIONS

USE EASY FUNCTIONS TO UNDERSTAND HARDER ONES

VI

• CAN YOU THINK OF A WAY TO SEE THAT $\lim_{x \to 0} e^{x} = 1$ (WITHOUT USING A CALCULATOR OR THE FACT THAT $e^0 = 1$)?

END

Q: IF $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$ CAN WE USE THE RULES ESTABLISHED TO FIND $\lim_{x \to a} f(x) - g(x)$?

$$\lim_{x \to a} f(x) + (-1) \cdot g(x) = \lim_{x \to a} f(x) + (-1) \cdot \lim_{x \to a} g(x)$$

• $\boxed{\text{WHAT IS } \lim_{x \to c} \frac{1}{x} =} \frac{1}{c}$

WHEN $c \neq 0$, WHEN $c = 0$ PROBLEM

THM

IF $\lim_{x \to a} f(x) = l$ AND $l \neq 0$

THEN $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l}$ (3)

WE KNOW HOW TO ADD (1) MULTIPLY (2) & DIVIDE (3) (ONLY IF NON-ZERO) LIMITS

WHAT CAN WE DO NOW? VII

- $\lim_{x \to a} \left(\frac{1}{x^3+4}\right) = \frac{1}{a^3+4}$

- $\lim_{x \to a} \left(\frac{1}{x^3-4}\right) = \frac{1}{a^3-4}$ $\boxed{\text{NEED } a \neq \sqrt[3]{4}}$

Q: IF $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$

CAN WE USE OUR RULES TO FIND

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}} = \lim_{x \to a} f(x) \cdot \frac{1}{g(x)}$$

$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{1}{g(x)} = l \cdot \frac{1}{m} = \frac{l}{m}$$

$\boxed{\text{ONLY IF } m \neq 0 \text{ !!!!}}$

- $\lim_{x \to a} \left(\frac{x^3-4}{x^2+1}\right) = \frac{\lim_{x \to a} (x^3-4)}{\lim_{x \to a} (x^2+1)} = \frac{a^3-4}{a^2+1}$

Q: WHY IS THIS OK??

- $\lim_{x \to 2} \left(\frac{x^2-4}{3x-6}\right)$ CAN I

DO THE SAME THING

$\boxed{a^2-b^2 = (a+b)(a-b)}$ KEY IDEA FROM HIGHSCHOOL

TECHNIQUE: FACTOR & CANCEL (VIII)

$$\lim_{x \to 2} \left(\frac{x^2-4}{3x-6} \right) = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}}$$

WHY CAN WE CANCEL?

IF $x=2$ WOULD HAVE A PROBLEM

$$= \frac{\lim_{x \to 2} (x+2)}{\lim_{x \to 2} 3} = \frac{4}{3}$$

OR

$$= \lim_{x \to 2} \left(\frac{1}{3}\right) \cdot \lim_{x \to 2} (x+2) = \frac{4}{3}$$

SOMETIMES CAN GET AROUND THE PROBLEM THAT DENOMINATOR'S LIMIT GOES TO ZERO

- $p(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x^1 + b_0$

$q(x) = c_m x^m + \dots + c_1 x^1 + c_0$

WITH $q(a) \neq 0$

THEN $\boxed{\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}}$

NOTICE USING ONLY $\lim_{x \to a} c = c$ & $\lim_{x \to a} x = a$

WE HAVE FOUND LIMITS OF ALMOST ALL RATIONAL FUNCTIONS!

POLYNOMIAL

- $\lim_{x \to 3} \sqrt{x} = \sqrt{3}$

LOOK IN BOOK RULES FOR $f(x)^{\frac{m}{m}}$

TECHNIQUE: USE CONJUGATES

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{(\sqrt{x}-1)}{(x-1)} \cdot \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{(x-1)}{(x-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

WHY CAN WE CANCEL ?!

$a^2-b^2=(a+b)(a-b)$ STRIKES AGAIN... BACKWARDS.

CHALLENGES: • WHAT ARE

$\lim_{x \to 0^+} e^{1/x}$ & $\lim_{x \to 0^-} e^{1/x}$ & $\lim_{x \to 0} e^x$

BONUS MARK

- $f(x) = \begin{cases} 1 & \text{IF } x \text{ RATIONAL} \\ 0 & \text{IF } x \text{ IRRATIONAL} \end{cases}$
- $g(x) = \begin{cases} x & \text{IF } x \text{ RATIONAL} \\ 0 & \text{IF } x \text{ IRRATIONAL} \end{cases}$

IS THERE A REAL # r SUCH THAT

$\lim_{x \to r} f(x)$ EXISTS? $\lim_{x \to r} g(x)$ EXISTS?

READ SECTION 2.6