A first excursion

The purpose of the first lectures outlined below is two-fold. The first is to introduce functions as our primary mathematical tool for expressing relationships between quantities. The second is to make a clear break between high school and university mathematics, laying the foundations for active learning throughout the term. Functions provide a great opportunity for this because it is a topic the students are very familiar with. As such, in principle, they are ready to be exposed to it in unfamiliar ways.

Functions. *What is a function?* this may elicit immediate answers from some students but, with enough encouragement, most of them will have an opinion (albeit surprisingly vague) about what a function is. This is a great opportunity to question them further: *Why does their definition make sense? Why would someone define a function in such a way? Why would someone bother defining a function at all? Who cares?*

Perhaps define a function f to be a rule which assigns to each real number x in its domain a unique real number $f(x)$. *Why would we want to require that the output be unique?* A few simple examples should reassure them that this resembles the 'formulaic' definition they are likely familiar with. It could also be gently indicated at this point that piecewise functions are covered by this definition.

What is a real number? What is a rational number? Another great point of discussion for which students will surely have opinions. Without dwelling too deep into it, the idea is to help them come to their senses and realize that they actually don't know what either of these two things are. Once they feel a bit more confident about the two concepts, they could be hit with

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{1}$$

Is this a function? Is there a "formula" that describes it? The purpose of such a discussion is to destabilize rather than comfort the students; making them realize that they will be expected to loosen their senses, unlearn what they have learnt and relearn everything in an active rather than passive manner.

This is a good moment to pause and reflect on the approach we have taken to define functions, perhaps addressing some points raised by the students at the beginning of the class. *Can we "see" a function? Why would we want to visualize a function? How could we extract something visual from our input/output point of view?* Students can be guided to realize that the inputs and outputs of a rule are merely ordered pairs and that we can think of a function as the (perhaps infinite) collection of ordered pairs specifying its outputs on all of its inputs. From here, the mental leap is to see that this slight change of perspective allows us to graph a function because ordered pairs can be plotted in a cartesian plane. We can see a whole (nice) function.

In order to fully understand the transition between these two points of view, we need a way to cast our definition of a function in terms of ordered pairs. With a bit of prompting (and allowing them to discuss among themselves if need be) they will eventually stumble upon the correct formulation: a function is a collection of ordered pairs of inputs and outputs with the special property that if (x, y) and (x, z) are in the collection, then $y = z$. *What does this special property of ordered pairs correspond to in a graph?* The vertical line test!

We now have a local (rule) and global (graphical) perspective on functions. *Are they really that different? Is one of them better than the other? Does it depend on the context?* A good instance where the local approach has a serious advantage is when we want to make sense of function composition. Students often seem a bit shaky on this topic so it is worth reviewing what it means and why it is useful. Armed with this concept a natural question arises: *Can we always "undo" the effect of a function? If so, how?* It will eventually occur to students that the only possible way to do this would be to swap the inputs and outputs of a given function to find one that "undoes" its effect. Trying to do this in concrete examples, they will quickly come to the realization that this does not always result in a function. Identifying the problem will lead them to the definition of a one-to-one function. They should be prompted to interpret this definition graphically and realize that they have discovered the horizontal line test. They should also be prompted to articulate the relationship between the graph of a function and that of its inverse. *Can we find an invertible function that coincides with its inverse?* In this context, the global graphical approach provides invaluable intuition.

Many students will not have come to understand yet that the true power of mathematics lies in its ability to simultaneously approach concepts from several different points of view. This is the first (of many) places to explicitly indicate this to them. It can also be good to emphasize that nothing in this course will be random, everything happens for a reason. The question they should constantly be asking themselves is not *"what am I doing?"* but rather *"why am I doing this?"*.

Exponentials and Logarithms. The big idea of calculus is to use "easy" functions to understand "hard" ones. *What could we mean by easy?* The best case scenario is usually that $f(x) =$ simple formula in terms of x. *What is the simplest possible formula we could think of?* Probably a constant function. *What is the second simplest formula we could think of?* Probably a linear function. In our context, "easy" will generally mean polynomial and our two favourite easy functions will be constant and linear ones. In fact, one might say that the big idea of calculus is to use lines to understand more complicated shapes. *What could we mean by hard?*

Probably some kind of exponential function (or trigonometric functions.. but we'll slip them under the rug for now). *Why would we need to care about such functions? Can we make sense of the formula* $f(x) = 2^x$*? Is this a function? What is its domain? What is the value of* $2^0, 2^{-1}, 2^{-n}, 2^{1/n}$*? What about* $2^{\sqrt{2}}$ *or* 2^π*?* Intuitively, we would like the domain of 2^x to be all real numbers but we can't seem to find an easy way to do that.

A magical number (that we will encounter again and again) with an infinite non-repeating decimal expansion $e = 2.718...$ saves the day! It has the mysterious property that raising it to the power of x coincides with an "infinite polynomial":

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots + \frac{1}{k!}x^k + \ldots \tag{2}$$

What could this possibly mean? How could we possibly make sense of such an expression? Students will most likely point out that we could plug in values on both sides an check that they match up. This prompts the question of evaluating the infinite polynomial at a point. Heuristically this can be done by evaluating successive partial sums, say at $x = 1$, and observing that it seems to "converge" to the given value of e (somewhat surprisingly, many students would be ready to accept this as a complete justification!). *How could we push this idea further to take into account all points at once?* Students should eventually realize that they could in fact plot the graphs of successive partial sums

$$f_0(x) = 1,\ f_1(x) = 1 + x,\ f_2(x) = 1 + x + \frac{1}{2}x^2,\ \ldots \tag{3}$$

to see that the graphs appear to "converge" to a plausible candidate for the function $f(x) = e^x$ (great opportunity for some computer experimentation).

At this point we should suspend our disbelief and suppose all of this makes sense, i.e., that we can evaluate the right hand side for all real numbers and that the limiting graph of the polynomials is what we expect. The question remains: why does the existence of such an expression save the day when it comes to taking powers like 2^x? So far, we don't have a complete answer but we have made progress: we know how to add and multiply all real numbers so we can evaluate the right hand side of the equation for any such number. This means that the domain of e^x is $\mathbb{R}$. We may also be willing to accept that $f(x) = e^x$ satisfies the properties you would expect to hold such as

$$f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1}e^{x_2} = f(x_1)f(x_2). \tag{4}$$

Put in a somewhat less familiar (but equivalent) way, this says that f is a function which turns addition into multiplication!

Look at the graph of $f(x) = e^x$. *What are its domain and range? Is this function invertible?* Yes! (vertical line test) but we don't have an explicit formula for its inverse. Nevertheless, since this function is of utmost importance, its inverse gets a special name:

$$g(y) = \ln(y). \tag{5}$$

Why are we using the variable y instead of the variable x? At this point, we know essentially nothing about $\ln(y)$. *Can we use properties of e^x to deduce properties of $\ln(y)$?* Remember that a function and its inverse "undo" each other's effect, in other words:

$$g(f(x)) = x \text{ and } f(g(y)) = y. \tag{6}$$

There is a bit of a yin and yang thing going on here and these identities are the key to understand the fundamental relationships between a function and its inverse, we will use them again and again. They can be thought of as a symmetry where any property of f has a "mirror" property in g. Spelling them out for exponential and logarithmic functions we obtain

$$\ln(e^x) = x \text{ and } e^{\ln(y)} = y. \tag{7}$$

Given that $e^{x_1+x_2} = e^{x_1}e^{x_2}$, can we identify the mirror property for $\ln$*?* Well,

$$\begin{aligned} e^{\ln(a)+\ln(b)} = e^{\ln(a)}e^{\ln(b)} &\implies \ln(e^{\ln(a)+\ln(b)}) = \ln(e^{\ln(a)}e^{\ln(b)}) \\ &\implies \ln(a) + \ln(b) = \ln(ab). \end{aligned} \tag{8}$$

Here, the last equality follows by recognizing the identity function written in a funny way. Indeed, we can always think of applying $f \circ g$ or $g \circ f$ as "doing nothing" when f is an invertible function with inverse g. This is entirely analogous to "adding zero" when completing the square in high school.

Given that $(e^{x_1})^{x_2} = e^{x_1 x_2}$, can we identify the mirror property for $\ln$*?* Well,

$$\begin{aligned} (e^a)^{\ln(b)} = e^{a\cdot\ln(b)} &\implies \ln(e^{a\ln(b)}) = \ln((e^{\ln(b)})^a) \\ &\implies a\ln(b) = \ln(b^a). \end{aligned} \tag{9}$$

Students should investigate other mirror properties on their own but, most importantly: *Can we now make sense of $h(x) = 2^x$? How could we take the "unknown" quantity 2^x and rewrite it in terms of things we understand?* This might require some discussion or hints but

$$h(x) = 2^x = e^{\ln(2^x)} = e^{x\ln(2)}. \tag{10}$$

Here, since we know how to make sense of $\ln(2)$, we know how to make sense of $x\ln(2)$. Moreover, since we know how to make sense of $e^{\text{some power}}$, we know how to make sense of $e^{x\ln(2)} = 2^x$. This is a common theme in mathematics, reinterpreting something we do not understand in terms of something that we do understand. *Was there anything special about the number 2?*