A first look at calculus

The purpose of the lectures outlined below is to illustrate a possible transition from the world of continuous functions into the differentiable world of calculus.

The Intermediate Value Theorem. Having spent some amount of time exploring the notion of continuity and building up a "zoo" of continuous functions (students should be encouraged to have a collection of "favourite" examples they get to know inside-out including polynomials, e^x, $\ln x$, $x^{1/2}$, $x^{1/3}$, $|x|$ etc.), the big question remains: *why do we care about continuous functions?*

Students should try to formulate an opinion of their own. After all, the only way to truly learn something is to care about it and the only way to care about something is to have our own reasons for doing so. One way to help them out here is to prompt them with some simple true or false questions (which may have been asked without definite answers in a previous lecture) such as:

(1) You were once exactly three feet tall.
(2) At some point since you were born, your weight in pounds was equal to your height in inches.
(3) The polynomial $p(x) = x^3 - 3x + 1$ has a root in $[0, 1]$.
(4) The polynomial $p(x) = x^3 - 3x + 1$ has two roots in $[0, 2]$.

These can lead to interesting class discussions about the nature of continuous functions and reinforce the idea that they are everywhere in our day-to-day lives. Having let the students struggle a bit, it is a good idea to remind them here that when introducing continuous function we were intuitively trying to rule out "abnormal" behaviour such as breaks or jumps in their graphs ensuring that we could draw them without "lifting our pen off the page". However, the abstract definition we cooked up was rather local in nature. Trying to reconcile the two naturally leads to the most basic answer to our opening question: the Intermediate Value Theorem. To begin, one might state it in its simplest form as follows:

If f is a continuous function on $[a, b]$ and $f(a) < 0 < f(b)$ then there is some $a < c < b$ such that $f(c) = 0$.

or, in language that the students will immediately understand:

The graph of a continuous function over a closed interval starting below the horizontal $x-$axis and ending above it must cross the axis at least once.

The hardest part about this theorem from the student's point of view is probably to realize that it actually is not obvious at all and that it embodies a mysterious property of the real numbers: the existence of least upper bounds. The other key thing to point out here is that this is saying something about the global behaviour of a continuous function on a closed interval based only on knowledge about its behaviour near points; this is our first local-to-global theorem!

Having absorbed the statement and some of the meaning behind this theorem, the class should be able to address the following questions: *Is there anything special about* 0 *(or the horizontal* $x-$*axis) in the statement of the theorem?* With a bit of prompting they should be able to reduce the general case to the statement given above. *Why do we require* f *to be continuous?* They will probably be able to answer this one with a heuristic example. This could be a good place to introduce the sign function. *Why is the Intermediate Value Theorem a big deal?* Amongst other things, we can finally settle the four true-or-false questions raised above!

The Derivative. This is where precalculus ends and the powerful ideas of calculus begin. As we have probably gathered by now, general functions

$$f(x) = \begin{cases} e^x & \text{if } x \in \mathbb{Q} \\ \sin(\frac{1}{x}) & \text{if } x \notin \mathbb{Q} \end{cases} \tag{1}$$

are too wild to be understood completely. Continuous functions such as $g(x) = |x|$ are nicer, but others such as

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{2}$$

can still be hard to tackle. We want something even *better*. Here, we remember that the goal of calculus is to use easy polynomial functions to understand harder ones. Accordingly, what we seek is a large nice class of functions for which this is possible.

As a first step, we might wish to use the easiest polynomials we know (linear ones) and try to understand functions in terms of tangent lines to their graphs. Drawing a smooth graph and a family of tangent lines would seem to indicate that they do capture some of the behaviour of the function. *What could we possibly mean by this?*

To put this on a firm footing we should probably start by making sure we know what we mean when we talk about tangent lines. Here is an enlightening example: at $(0, 0)$, does the graph of $f(x) = |x|$ have

(a) a tangent line $y(x) = 0$?
(b) infinitely many tangent lines?
(c) no tangent line?
(d) two tangent lines $y(x) = x$ and $y(x) = -x$?

The interesting point to be made here is that all of the answers are plausible and this highlights our need for precise definitions avoiding any ambiguity (which may not be so clear to students from the get-go). *How could we come up with a suitable and precise definition for a tangent line?*

As usual, we must start with what we know and work our way towards the concept we are after. Given a function f, we can make sense of many lines through the point $(a, f(a))$: the secants through other points of the form $(a+h, f(a+h))$. Thinking visually, our naive conception of a tangent line would seem to coincide with the "limit" of such secants as $h \to 0$. Now, we don't know how to talk about a "limit" of lines, but we do know how to make sense of the limit of the slopes of such lines:

$$\lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}. \tag{3}$$

According to this philosophy, the slope of a tangent line to the graph of f at $(a, f(a))$ "should" coincide with this expression. If this limit exists, we *call* it the derivative $f'(a)$ of f at a and say that f is differentiable at a. In this case, we can *define* the tangent line to the graph of f at $(a, f(a))$ to be the line through this point with slope $f'(a)$. (This is a good place to revisit the multiple choice question above.)

The notation we chose for the derivative is reminiscent of that of a function. This is no accident: if f is differentiable on an interval we can think of f' as a function which takes points of this interval as inputs and outputs the slope of the tangent line to the graph of f at these points:

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \tag{4}$$

The derivative, if it exists, encodes the slopes of all the tangent lines to the graph of a function. Accordingly, it should be telling us something about the shape of this graph but we'll hold that thought for now and try to compute it for our favourite functions.

As usual, we should start with the easiest possible functions: $f(x) = c$, $g(x) = x$ and check that the answers match our intuition. We could then move on to harder functions such as $f(x) = x^3$ and $g(x) = x^{1/3}$. Here, it is insightful to sketch the graph of these functions first and discuss whether they "should" have tangent lines (in particular at the origin). Although we can see a vertical "tangent" line at the origin for $g(x) = x^{1/3}$, the corresponding limit does not exist. We made our definition, now we have to live with it. *Why does it make sense in our context to rule out vertical "tangent" lines?* (We want the derivative to be a function which encodes the shape of the graph of f.) This is a good place to superimpose the graphs of the functions we just discussed with the graphs of their derivatives to help students "see" what's going on. To complete the cycle, it is also helpful to figure out what happens for $h(x) = |x|$.

In light of our initial motivation, a natural question arises: *Is there a relationship between continuous functions and differentiable functions?* Some discussion (noticing in particular that $f(x) = x^{1/3}$ is continuous but not differentiable everywhere) may lead the class to the more refined question: *Are all differentiable functions continuous?* or *If f is differentiable at a, does it imply that f is continuous at a?*

Having established the above in the affirmative, this is a great place to do a bit of logic since some students will have never heard of implications or contrapositives before. Usually a few examples suffice to get the general idea across. The following question is a good litmus test to check that they understood the concept: Given that

$$\text{if } f(x) \text{ is a polynomial then it must be continuous,}$$

which of the following is true?

(a) If $f(x)$ is not continuous then $f(x)$ is not a polynomial.
(b) If $f(x)$ is continuous then it is a polynomial.
(c) If $f(x)$ is not a polynomial then it is continuous.