A little bit of economics

The goal of the lectures outlined below is to tie in some of the material previously covered with economics. The first offers an alternative approach to elasticity while the second highlights an unexpected link between e and compound interest.

Price Elasticity of Demand. When confronted with a new concept it can often be enlightening to go back to the original source. In his Principles of Economics (1890) Alfred Marshall states that:

> "The elasticity (or responsiveness) of demand in a market is *great or small* according as the amount demanded increases *much or little* for a given fall in price, and diminishes *much or little* for a given rise in price."

This original definition is a great source of class discussion: *What does he mean by great or small and much versus little? How would you make sense of Marshall's statement? If demand increases by* 10 *units for a fall in price of* \$1*, is this much or little?* Since most students will have had some prior exposure to elasticity, a class-wide discussion should eventually converge upon the following kind of definition: Elasticity is the measurement of how "responsive" an economic variable (in our case demand 'q') is to a change in another (in our case price 'p'). Here, the way to make sense of this responsiveness (resp. change) is to compare it to the initial value of the corresponding variable. In other words, price elasticity of demand measures the relative change in demand implied by a relative change in price. *Can you think of a situation where a small relative change in price implies a large relative change in demand? Can you think of a situation where a large relative change in price implies a small relative change in demand? What are the factors at play in each situation you described? Which of the situations would you call elastic or inelastic?*

Having established that our main players are the relative changes $\frac{\Delta q}{q}$ and $\frac{\Delta p}{p}$, we can now try to put elasticity on a firmer mathematical footing by seeing precisely how these quantities are related. Thinking of p as the independent variable and $q = q(p)$ as the dependent variable, for any fixed change Δp there is some number $E(p)$ for which

$$E(p) \cdot \frac{\Delta p}{p} = \frac{\Delta q(p)}{q(p)} = \frac{q(p + \Delta p) - q(p)}{q(p)}. \tag{1}$$

It should be highlighted that what we have just implicitly defined is a function $E(p)$. *Should $E(p)$ be positive or negative?* Thinking about the law of supply and demand (bearing in mind that economists often define this quantity in absolute value which leads to some sign discrepancies) should lead them to the right answer. *What is $E(p)$ measuring?* They should observe that it measures how strongly a relative change in p impacts a relative change in $q(p)$ and, thus, offers us a first insight into mathematically quantifying price elasticity of demand:

$$E(p) = \left(\frac{\Delta\, q}{q}\right) / \left(\frac{\Delta\, p}{p}\right) = \%\text{ change in demand}/\%\text{ change in price.} \tag{2}$$

At this point we should remind ourselves that we are in a calculus class. *Can we interpret $E(p)$ using calculus?* Proceeding naively, we may observe that

$$E(p) = \left(\frac{\Delta\, q}{q}\right) / \left(\frac{\Delta\, p}{p}\right) = \frac{p}{q} \cdot \frac{\Delta\, q}{\Delta\, p} \tag{3}$$

and taking a limit as $\Delta\, p \to 0$ we obtain

$$E(p) = \frac{p}{q} \cdot \frac{\Delta\, q}{\Delta\, p} \xrightarrow{\Delta\, p \to 0} \frac{p}{q} \cdot \frac{dq}{dp}. \tag{4}$$

In light of this approximation, we can now choose to *define* (just like we chose to define tangent lines a certain way in light of our geometric reasoning) the price elasticity of demand as

$$\varepsilon(p) := \frac{p}{q} \cdot \frac{dq}{dp}. \tag{5}$$

One interesting thing to note here is that $\varepsilon(p)$ has the same intuitive meaning as $E(p)$ even if this is no longer so clear from its defining formula! *What is the advantage of $\varepsilon(p)$ over $E(p)$?*

Although price elasticity of demand is a multifaceted topic, our main application of the concept will be to understand how $\varepsilon(p)$ affects marginal revenue. On a first glance, this might seem a bit weird. *Can you give an intuitive reason why this ought to be the case?* Let's recall that if we think of p as the independent variable and $q = q(p)$ as the dependent variable then our expression for the revenue is $R(p) = p \cdot q(p)$. From a business point of view, we often also care about its derivative, the marginal revenue $R'(p)$. *Why? Can we compute this derivative? How?* By the product rule or implicit differentiation we obtain

$$R'(p) = q(p) + p \cdot \frac{dq}{dp} = q(p)\left(1 + \frac{p}{q}\frac{dq}{dp}\right) = q(p)\,(1 + \varepsilon(p)) \tag{6}$$

and we see that comparing $|\varepsilon(p)|$ to 1 tells us the sign of $R'(p)$. *Why should we care about this sign? What happens if $|\varepsilon(p)| < 1$, $|\varepsilon(p)| = 1$ and $|\varepsilon(p)| > 1$?*

Why Should You Care About Exponentials? Suppose you invest \$1 into an account with an interest rate of 100% per year, compounded n times per year (this concept will not be completely familiar to all students so it's worth specifying that this means we add $\frac{100}{n}\%$ of the *current* value to the account n times per year). *What is the value of your account after one year if you compound interest:*

- $n = 1$ times?
  $\frac{100\%}{1} = 1$ and we compound *once* so the value after one year is

$$1 + 1 \cdot 1 = \$2.$$

- $n = 2$ times?
  $\frac{100\%}{2} = \frac{1}{2}$ and we compound *twice* so the value after one year is

$$1 + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \left(1 + \frac{1}{2} \cdot 1\right) = \left(1 + \frac{1}{2}\right) + \frac{1}{2}\left(1 + \frac{1}{2}\right) = \left(1 + \frac{1}{2}\right)^2 = \$2.25. \tag{7}$$

  (Students could be guided to discover the fractal pattern underlying this computation leading to the neat formula.)
- $n = 3$ times?
  $\frac{100\%}{3} = \frac{1}{3}$ and we compound *thrice* so the value after one year is

$$\begin{aligned} &1 + \frac{1}{3} + \frac{1}{3}\left(1 + \frac{1}{3}\right) + \frac{1}{3}\left[1 + \frac{1}{3} + \frac{1}{3}\left(1 + \frac{1}{3}\right)\right] \\ &= \left(1 + \frac{1}{3}\right)^2 + \frac{1}{3}\left(1 + \frac{1}{3}\right)^2 = \left(1 + \frac{1}{3}\right)^3 = \$2.37 \end{aligned} \tag{8}$$

Higher compounding frequency leads to higher investment value after one year!

- $n = 365$ times?
  By now the students should be able to answer this at once:

$$\left(1 + \frac{1}{365}\right)^{365} = \$2.71. \tag{9}$$

- n times?

$$\$\left(1 + \frac{1}{n}\right)^n. \tag{10}$$

Knowing that higher compounding frequency leads to higher investment value at the end of the year, one might naturally wonder if we could make our return arbitrarily large by compounding more and more. *What happens if we let $n \to \infty$?* In this case we say that the interest is compounded continuously and our return after a year is

$$\$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n. \tag{11}$$

What do you think the limiting value should be? Typical answers may include ∞, 3 or even e although most students have probably never seen this before. *Can we evaluate this limit? What is your go-to method for handling annoying powers?* We usually start out by using our favourite pair of a function and its inverse

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} \tag{12}$$

along with the fact that, for continuous functions, the limit of a composition is the composition of the limits

$$\lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} = e^{\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n}. \tag{13}$$

Focusing in on the logarithm, the students can now be guided step-by-step (providing some heuristic motivation for each trick along the lines of *what is the simplest non-trivial manipulation you could do?*):

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}. \tag{14}$$

At this point, it can be helpful (in an attempt to simplify the situation) to rewrite the expression using $h := \frac{1}{n}$. Observing that $n\to\infty$ if and only if $h\to 0$, we obtain a new expression:

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{h\to 0}\frac{\ln(1+h)}{h}. \tag{15}$$

Have we made progress? Why or why not? Can you spot a derivative? With a bit of prompting, students should eventually realize that $\ln(1) = 0$ so our expression is simply the derivative of $\ln(x)$ evaluated at $x = 1$, i.e.,

$$\lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \frac{d}{dx}|_{x=1}\ln(x) = 1 \tag{16}$$

and we can safely conclude that:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n} = e^1 = e. \tag{17}$$

Most students will be very surprised (maybe even shocked) when they realize that this means their return after a year would be of \$$e$. As an added benefit, the students will also realize that they can handle quite elaborate limits by using the concepts they have previously encountered in creative ways.