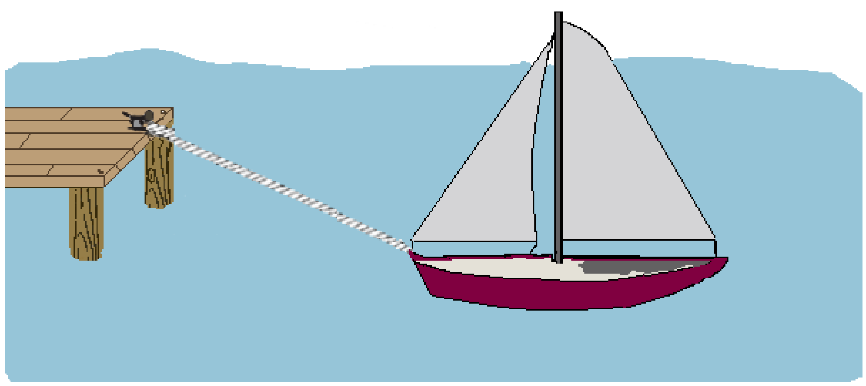
Problem Solving

At this point in the course, students are beginning to have a good understanding of the behaviour of differentiable functions. The purpose of the lectures alluded to below is to return their attention to our initial motivation which was to *use* this understanding of differentiable functions in order to solve concrete problems.

Related Rates. It could probably be argued that the related rates problems students will face in such a course are often the trickiest. This is because in these problems, unlike for other topics, the very first step is the most conceptually demanding. It is therefore crucial that students develop a solid algorithmic approach when it comes to setting them up. Luckily, the very nature of these problems makes them easy to discuss as a class; students should be encouraged to use their intuition to try and predict answers before talking about their guesses with their classmates and, lastly, use mathematics to verify whether their intuition was correct!

For instance, one could begin with the following problem from the Good Questions Project at Cornell:

A boat is being drawn close to a dock by pulling a rope using a winch as shown[1]. How is the rate at which the rope is being pulled related to the rate at which the boat approaches the dock?



The first thing to notice about this problem as an instructor is that it doesn't involve any numbers. The reason behind this is to emphasize that the main point is the setup of the problem as opposed to number crunching, thereby allowing students to use their intuition freely without numerical constraints. In order to facilitate a class discussion and motivate students to formulate a guess even if they aren't sure, they could also be given three possibilities to choose from:

[1]The author would like to thank Natanz Handy for drawing the figure.

(a) They are constant multiples of each other.
(b) They are equal.
(c) It depends on how close the boat is to the dock.

It is hard enough to guess the correct answer to this question that students should end up divided in two or three camps. Now that they're invested, having opinions of their own, they will be more eager to prove each other right or wrong using the sound footing of mathematics. A relatively straightforward sequence of guiding questions should then lead them to discover that, in fact, it depends on how close the boat is to the dock. For instance, this could be done via the following steps which could in turn be used as a template for the students when they later try to solve such problems on their own:

- *Can you draw a diagram representing the situation at hand?*

  (This should be fairly straightforward for this particular problem though, in general, it can be a bit tricky.)
- *Can you label the relevant quantities? Which ones are constants? Which ones are variables, i.e., functions?*

  (It is critical that students understand the distinctions between the two in the context of a related rates problem. This seemingly innocuous step should not be underestimated.)
- *Can you find an equation relating the functions you are interested in?*

  (Students may need a hint to realize they must use the Pythagorean Theorem. It might also be pointed out that this step often involves spotting similar triangles.)
- *Can you implicitly differentiate your equation to obtain one relating the rates of change you are interested in?*

  (This will be fairly straightforward if the students have properly understood the inner workings of implicit differentiation.)

At this point, students should be feeling quite comfortable with this particular problem setup and this is a great moment to hit them with the following question: *Will the boat reach the speed of light before it reaches the dock?* Students will be shocked to discover that a simple manipulation of the equations they obtained answers the question in the affirmative. *Where did we go wrong!?* This should make for an interesting class discussion on the simplifications we often have to make in our toy models of situations and how they may affect the validity of our results. The reality of such issues should be no stranger to anyone familiar with recent economic meltdowns.

An Unconventional Optimization Problem. Students quickly grow tired of typical constrained optimization problems. Indeed, the appropriate answer to most questions about ladders leaning against fences is *who cares.* One way around this is to seek optimization problems that they have probably encountered before in their lives (and may still be encountering) without thinking about them in this way. A particularly interesting source of these "real" problems comes from online video games where players have to make decisions under various constraints to secure a victory. The following questions found on a Stack Exchange thread illustrate one way to exploit this as a pedagogical set up.

In the online PC game League of Legends, one of 67 million players per month assumes the role of a "champion" battling against other players in an arena. To maximize their chances of winning, players must use gold to buy Health and Armour in order to optimize their Effective health

$$E = \frac{H(100 + A)}{100} \tag{1}$$

when defending against physical damage.

Given that a player starts the game with 3600 *gold, Health costs* 2.5 *gold per unit and Armour costs* 18 *gold per unit, how much Armour and Health should they buy to survive as long as possible?* The expression for E isn't particularly complicated and, just like with the related rates problems, the emphasis here is on the set up. The thing students need to realize is that our target function depends on two variables H and A (as with most optimization problems). The key is to discover the (somewhat hidden) relationship between them ($2.5H + 18A = 3600$) which allows us to reduce it to a single variable problem we can handle. Although this won't be immediately apparent to the students since the problem isn't worded in the usual "maximize the volume of a box with fixed area" language, they should be left to discover it as a class.

Suppose at some point in the game your health (resp. armour) is down to 1080 *(resp.* 10*). If a teammate gives you* 720 *gold, how would you spend it to optimize your resulting health and armour?* The catch here is that students won't immediately realize this isn't the same problem as the one they just solved. With a bit of discussion amongst themselves, they should eventually discover that the function they need to optimize has changed and is now

$$E' = \frac{(1080 + H)[100 + (10 + A)]}{100}. \tag{2}$$