POSTED NOTE ABOUT MIDTERM ON PIAZZA

LAST TIME

INCREASING / DECREASING TEST → FIRST DERIVATIVE TEST

$f' > 0$ ON $(a,b) \Rightarrow f$ INCREASING ON (a,b)

$f' < 0$ ON $(a,b) \Rightarrow f$ DECREASING ON (a,b)

INCEPTION AKA CONCAVITY TEST → SECOND DERIVATIVE TEST

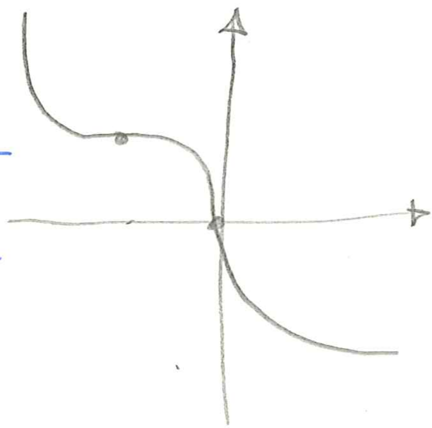
$f'' > 0$ ON $(a,b) \Rightarrow f$ CONCAVE UP ON (a,b)

$f'' < 0$ ON $(a,b) \Rightarrow f$ CONCAVE DOWN ON (a,b)

@ END OF CLASS 11AM

[Q] SKETCH GRAPH OF f WITH

- DOMAIN $(-\infty, +\infty)$
- $f(0) = 0$, $f(-1) = 1$
- $f'(x) < 0$ ON $(-\infty, +\infty)$
- $f''(x) > 0$ ON $(-\infty, -1)$
- $f''(x) < 0$ ON $(-1, 0)$
- $f''(x) > 0$ ON $(0, +\infty)$

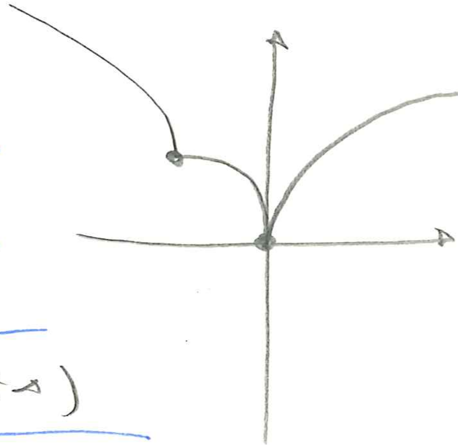


OK TODAY WE RE-DO THE

"SAME" QUESTION WITH A TWIST:

[Q] SKETCH GRAPH OF f WITH (II)

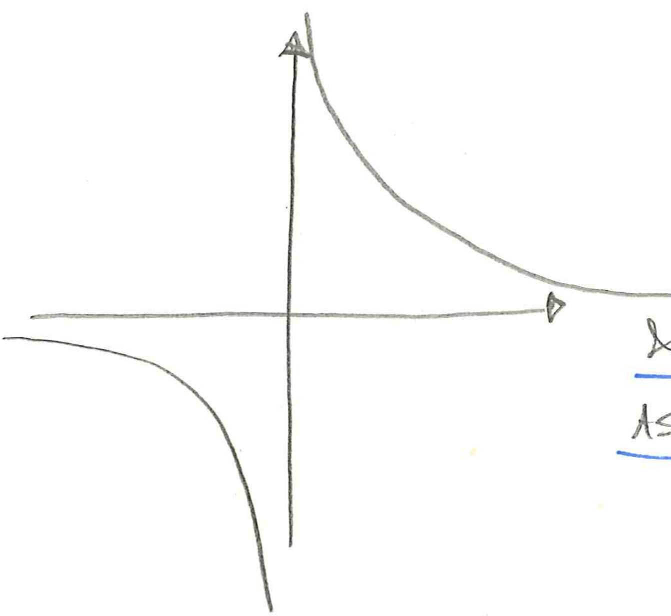
- DOMAIN $(-\infty, +\infty)$
- $f(0) = 0$, $f(-1) = 1$
- $f'(x) < 0$ ON $(-\infty, 0)$
- $f'(x) > 0$ ON $(0, +\infty)$
- $f''(x) < 0$ ON $(-\infty, +\infty)$



? DIDN'T SAY ANYTHING ABOUT DERIVATIVE EXISTING AT 0 & -1 !!!

TODAY: FINISH CURVE SKETCHING BY UNDERSTANDING ASYMPTOTES

ARCHETYPAL EXAMPLE: $f(x) = \frac{1}{x}$



HAS A VERTICAL ASYMPTOTE $x = 0$

& A HORIZONTAL ASSYMPTOTE $y = 0$

HOW DO WE KNOW IT LOOKS LIKE THIS?

USE OUR MACHINERY

$f'(x) = -\frac{1}{x^2}$ ⇝ DNE WHEN $\boxed{x=0}$

f HAS SINGLE CRITICAL POINT

$f''(x) = \frac{-(-2)}{x^3} = \frac{2}{x^3}$ ⇝ DNE WHEN $\boxed{x=0}$

f' HAS SINGLE CRIT

INTERVAL	f'	f
$(-\infty, 0)$	<0	DECREASING
$(0, \infty)$	<0	DECREASING

INTERVAL	f''	f
$(-\infty, 0)$	<0	CD
$(0, \infty)$	>0	CU

WHAT ABOUT FINDING THE ASYMPTOTES!?

DEFINITION

IF $\lim_{x \to a^+} f(x) = \pm\infty$

OR $\lim_{x \to a^-} f(x) = \pm\infty$

USUALLY HAPPENS WHEN DENOMINATOR VANISHES AT a

THEN WE SAY THE LINE $x = a$ IS A VERTICAL ASYMPTOTE OF f

IF $\lim_{x \to +\infty} f(x) = L$

THEN WE SAY f HAS A HORIZONTAL ASYMPTOTE $y = L$ AT $+\infty$

SIMILARILY, IF $\lim_{x \to -\infty} f(x) = M$

THEN WE SAY f HAS A HORIZONTAL ASYMPTOTE $y = M$ AT $-\infty$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \qquad \lim_{x \to -\infty} \frac{1}{x} = 0$$

EXACTLY WHAT WE SUSPECTED

V

RECALL

$\lim_{x \to a^+} f(x) = L$ IF WE CAN MAKE $f(x)$ AS CLOSE AS WE LIKE TO L BY REQUIRING x TO BE SUFFICIENTLY CLOSE TO BUT STRICTLY LARGER THAN a

(THINK $x = a + 0.0000000000\ldots00001$)

SO $\lim_{x \to a^+} f(x) = +\infty$ IF CAN MAKE $f(x)$ AS BIG AS WE LIKE (AKA AS CLOSE AS WE LIKE TO $+\infty$) B/...

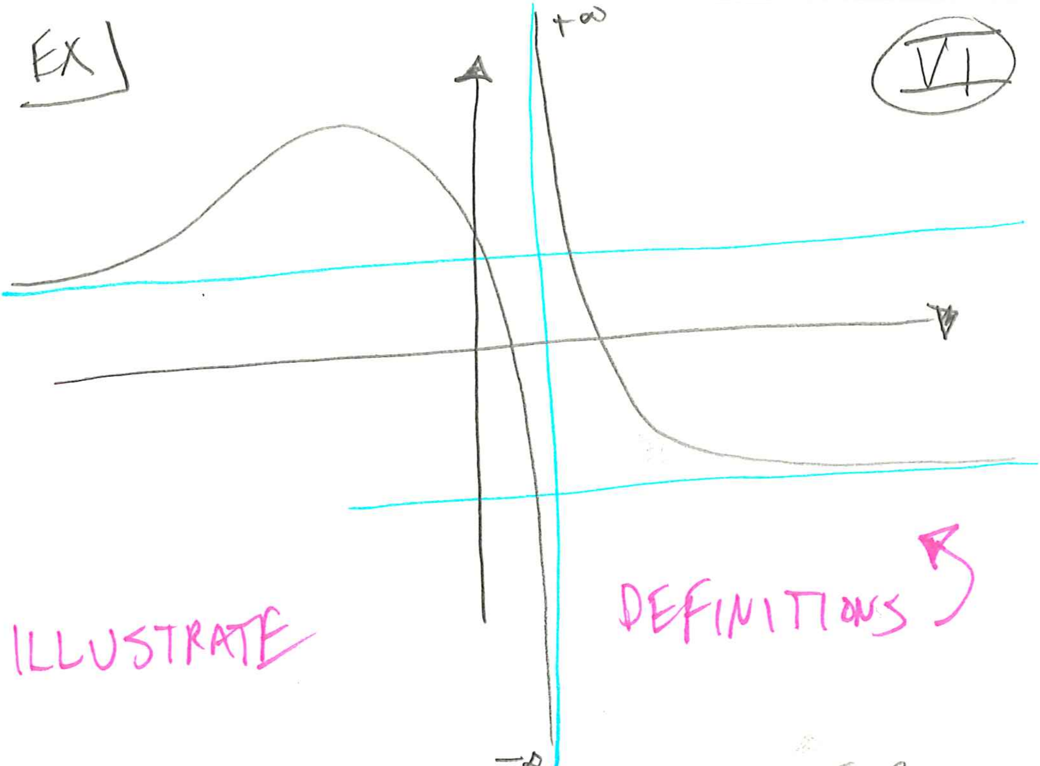
SIMILARILY $\lim_{x \to a^+} f(x) = -\infty$ IF CAN MAKE $f(x)$ AS SMALL AS WE LIKE B/... (AKA AS CLOSE AS WE LIKE TO $-\infty$)

LASTLY

$\lim_{x \to +\infty} f(x) = L$ IF WE CAN MAKE $f(x)$ AS CLOSE AS WE LIKE TO L BY REQUIRING x TO BE LARGER THAN SOME HUGE POSITIVE NUMBER (SMALLER THAN SOME ~~TINY~~ HUGE NEGATIVE NUMBER FOR $-\infty$)

EX

VI



ILLUSTRATE

DEFINITIONS

TYPICALLY, ONLY NEED TO DO THIS FOR POLYNOMIAL FRACTIONS (IN THIS COURSE)

THM IF $r > 0$ IS A RATIONAL NUMBER

THEN $\lim_{x \to +\infty} \frac{1}{x^r} = 0$

IF $r > 0$ IS A RATIONAL NUMBER FOR WHICH x^r IS DEFINED FOR ALL x

THEN $\lim_{x \to -\infty} \frac{1}{x^r} = 0$

WHY EXTRA CONDITION !?

EX)
$$f(x) = \frac{5x^3 + 3x - 2}{x^4 - x^2 + 4}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^3 \left(5 + 3/x - 2/x^3\right)}{x^4 \left(1 - 1/x^2 + 4/x^4\right)}$$

LET $h(x) = \frac{x^3}{x^4}$ & $g(x) = \frac{\left(5 + 3/x - 2/x^3\right)}{\left(1 - 1/x^2 + 4/x^4\right)}$

THEN
$$\lim_{x \to +\infty} h(x) = \lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \left(\frac{5 + 3/x - 2/x^3}{1 - 1/x^2 + 4/x^4} \right) = 5$$

$\boxed{\text{SINCE BOTH EXIST}}$ CAN USE LIMIT RULE

$$\Rightarrow \lim_{x \to +\infty} f(x) = \left(\lim_{x \to +\infty} h(x) \right) \left(\lim_{x \to +\infty} g(x) \right)$$

$$= 0 \cdot 5 = 0$$

CAN USE LIMIT RULES BUT HAVE TO BE CAREFUL

↳ $\lim_{x \to a} f(x) = +\infty$ IS SHORTHAND FOR THE LIMIT DNE BUT GROWS ARBITRARILY LARGE

$+\infty$ IS NOT A NUMBER!

GRAPHING A FUNCTION f

- FIND DOMAIN OF DEFINITION
- FIND f' & f'' & GET ALL INFO YOU CAN OUT OF THEM
- LOCATE VERTICAL & HORIZONTAL ASYMPTOTES
- FIND x & y INTERCEPTS

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$$\boxed{f(x) = \frac{x^2+5x+1}{x^2}} = 1 + \frac{5}{x} + \frac{1}{x^2}$$

DOMAIN: REAL LINE EXCEPT $x=0$

DERIVATIVES:

$$f'(x) = \frac{-5}{x^2} - \frac{2}{x^3} = \frac{-(5x+2)}{x^3}$$

$$f''(x) = \frac{10}{x^3} + \frac{6}{x^4} = \frac{(10x+6)}{x^4}$$

EASIER TO TELL SIGN vs COMPUTE

IX

CRIT POINTS : $x = 0 \quad x = \frac{-2}{5}$

POSSIBLE INFLECTION POINTS : $x = 0 \quad x = -\frac{6}{10} = \frac{-3}{5}$

INTERVAL	f'	f
$(-\infty, \frac{-2}{5})$	< 0	$\downarrow$
$(-2/5, 0)$	> 0	$\uparrow$
$(0, +\infty)$	< 0	$\downarrow$

INTERVAL	f''	f
$(-\infty, -3/5)$	< 0	$\cap$
$(-3/5, 0)$	> 0	$\cup$
$(0, +\infty)$	> 0	$\cup$

LOCAL MIN AT $-2/5$, INFLECTION POINT AT $-3/5$

ASYMPTOTES

X

CHECK WHERE DENOMINATOR VANISHES

$$\lim_{x \to 0^+} \left(1 + \frac{5}{x} + \frac{1}{x^2}\right) = +\infty$$

$$\lim_{x \to 0^-} \left(1 + \frac{5}{x} + \frac{1}{x^2}\right) = +\infty$$

VERTICAL ASYMPTOTE

DIDN'T HAVE TO BE SAME

$$\lim_{x \to +\infty} \left(1 + \frac{5}{x} + \frac{1}{x^2}\right) = 1$$

$$\lim_{x \to -\infty} \left(1 + \frac{5}{x} + \frac{1}{x^2}\right) = 1$$

HORIZONTAL ASYMPTOTE

INTERCEPTS

$$x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{21}}{2} \rightarrow \begin{matrix} -4.8 \\ -0.2 \end{matrix}$$