LAST TIME

I

FOR x CLOSE TO a

HAVE LINEAR APPROX OF f AT a

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

HOW CLOSE TO a DOES x NEED TO BE?

IT DEPENDS HOW GOOD YOU WANT THE APPROXIMATION TO BE

THE ERROR AT x: $|R(x)| = |f(x) - L(x)|$

CAN BE BOUNDED

$$|R(x)| \leq \frac{M}{2} |x-a|^2$$

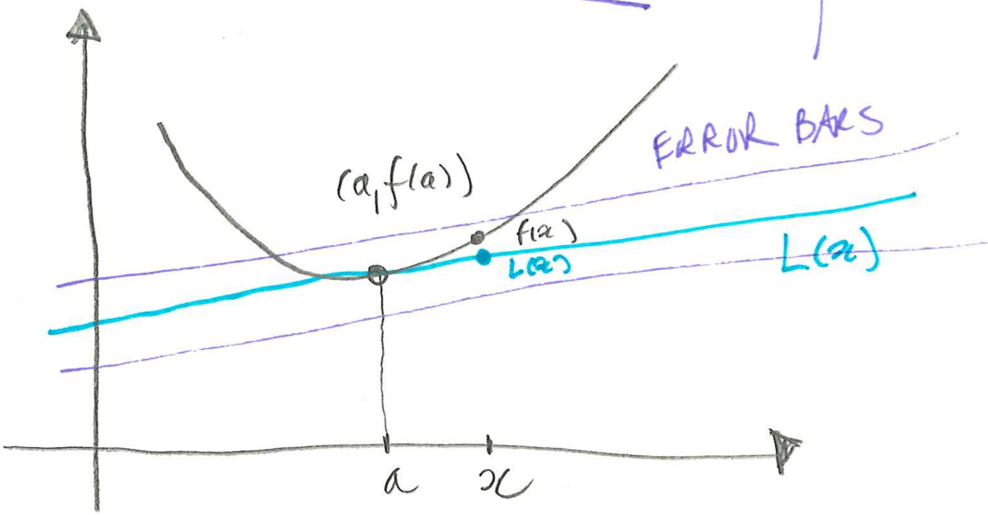
WHERE M IS ANY NUMBER THAT IS BIGGER THAN ALL VALUES $f''(c)$ FOR c BETWEEN a AND x

(IF YOU CAN FIND ACTUAL MAX, USE IT!)

THE MEANING OF $|R(x)|$ IS: (II)

$$L(x) - |R(x)| \leq f(x) \leq L(x) + |R(x)|$$

TRAPPED



[Q] WE KNOW $f(a) = L(a)$

EASY

WHAT ABOUT $f'(a)$, IS IT EQUAL TO $L'(a)$?

[A] $L(x) = f(a) + f'(a)(x-a)$

$\Rightarrow L'(x) = f'(a)$

$\Rightarrow L'(a) = f'(a)$ ✓

III

ANOTHER WAY TO LOOK AT LINEAR APPROX:

GIVEN DIFF $f(x)$, FIND A POLYNOMIAL $P(x)$ WHICH APPROXIMATES $f(x)$ NEAR $x=a$

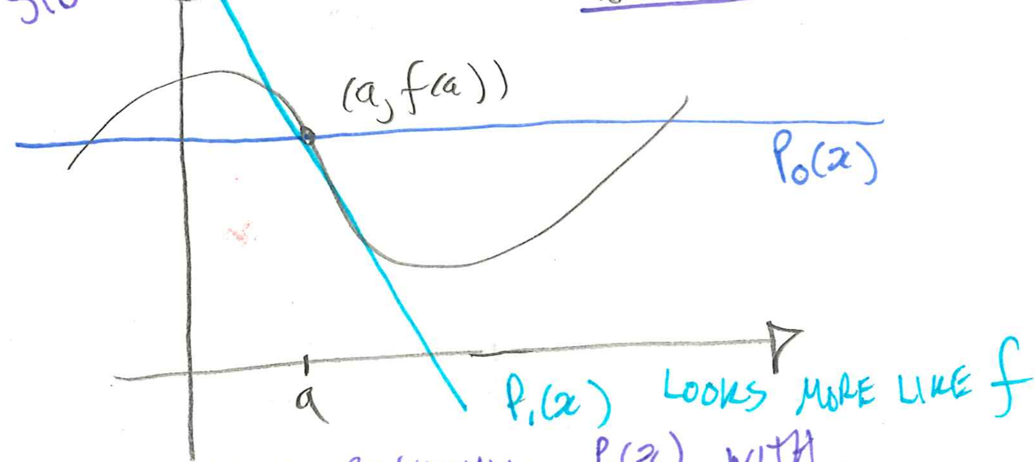
& (0) AGREES WITH f AT a

(ie $f(a)=p(a)$)

LAZYEST ANSWER: $P_0(x)=f(a)$ CONSTANT

THEN $P_0(a)=f(a)$ ✓

SIDE



FIND POLYNOMIAL $P_1(x)$ WITH

(1): $f(a)=P_1(a)$ AND $f'(a)=P_1'(a)$

LAZYEST ANSWER

[Q] WHAT IS MINIMAL DEGREE OF $P_1(x)$ HERE?

IV

$\Rightarrow \boxed{P_1(x) = c_0 + c_1 x}$

CONSTANTS

HOW DO WE FIND THEM? [Q]

$P_1(a) = f(a)$ & $P_1'(a) = f'(a)$

$P_1(a) = c_0 + c_1 a$

$P_1'(a) = c_1$

$\Rightarrow \boxed{c_1 = f'(a)}$

$\Rightarrow c_0 + f'(a)\cdot a = f(a)$

$\Rightarrow \boxed{c_0 = f(a) - f'(a)\cdot a}$

SO $P_1(x) = \underbrace{(f(a) - f'(a)\cdot a)}_{c_0} + \underbrace{(f'(a))}_{c_1} x$

$= f(a) + f'(a)(x-a)$

DEGREE ONE!

$\Rightarrow$ STUMBLE BACK ON LINEAR APPROX

[Q] WHAT NEXT?

FIND POLYNOMIAL $P_2(x)$ WITH

(2) $f(a) = P_2(a)$, $f'(a) = P_2'(a)$

& $f''(a) = P_2''(a)$

V

[Q] WHAT IS MIN POSSIBLE DEGREE OF $P_2(x)$?

$$\boxed{P_2(x) = C_0 + C_1 x + C_2 x^2}$$

[Q] HOW DO WE FIND C_0, C_1, C_2 ?

USE THE CONDITIONS TO GET (AFTER REARRANGING)

$$\boxed{P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2}}$$

[Q] WHY IS THERE A 2 ON BOT ?

CANCEL $(\ )^2$ COMING FROM DIFF.

$\boxed{\text{CAUTION}}$: THE $\frac{f''(a)(x-a)^2}{2}$ & $\frac{M|x-a|^2}{2}$ ARE RELATED ... BUT NOT THE SAME

WHY? $f''(a)$ IS VALUE AT a, M IS $\max_{a \le c \le x} f''(c)$

IN FACT THERE IS ANOTHER, MORE COMPLICATED BOUND FOR $f(x) - P_2(x)$

Ex] $f(x) = e^x \rightsquigarrow$ FWD $P_0(x)$, $P_1(x)$, $P_2(x)$ APPROX AT ZERO

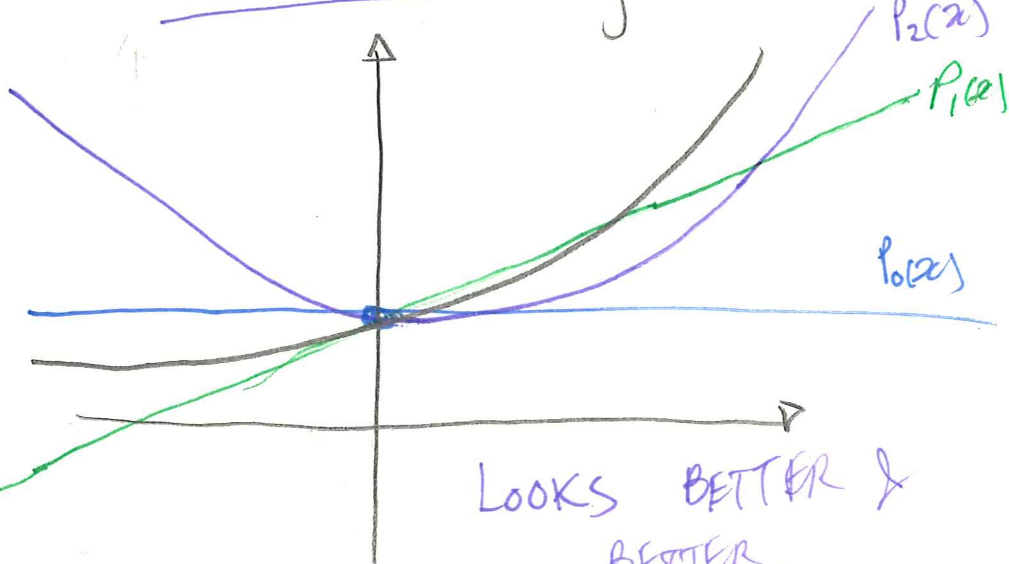
VI

DO IT!

$f'(x) = e^x \rightsquigarrow 1$

$f''(x) = e^x \rightsquigarrow 1$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$



LOOKS BETTER & BETTER..

& WE CAN KEEP GOING!

[Q] WHAT SHOULD $P_3(x)$ BE?

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2}$$

WHY IS IT NOT DIVIDED BY 3!?

VII

$$P_3'(x) = 0 + 1 + \frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2}$$

$$P_3''(x) = 0 + 0 + 1 + \frac{\cancel{2}x}{\cancel{2}}$$

$$P_3'''(x) = 0 + 0 + 0 + 1$$

EVAL AT ZERO.

THESE SOLUTIONS TO PROBLEM OF FINDING A POLYNOMIAL $P_n(x)$ WHICH AGREES WITH ALL DERIVATIVES UP TO n OF f

$$\begin{array}{ccccc} f(a), & f'(a), & f''(a), & f^{(3)}(a), \ldots, & f^{(n)}(a) \\ \| & \| & \| & \| & \| \\ P_n(a) & P_n'(a) & P_n''(a) & P_n^{(3)}(a) & P_n^{(n)}(a) \end{array}$$

AT A POINT a

ARE CALLED n-TH TAYLOR POLYNOMIAL APPROXIMATING f AT a.

$$P_n(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2\cdot 1}(x-a)^2$$

$$+ \frac{f^{(3)}(a)}{3\cdot 2\cdot 1}(x-a)^3 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

VIII

HOW GOOD ARE THESE APPROXIMATIONS?

THM LAGRANGE REMAINDER

SUPPOSE $f^{(n)}(x)$ IS DIFF

(ie $f^{(n+1)}(x)$ EXISTS)

IF $P_n(x)$ IS n-TH TAYLOR APPROX OF $f(x)$ AT a AND $R_n(x) = f(x) - P_n(x)$

THEN $$|R_n(x)| \leq \frac{M}{(n+1)(n)(n-1)\cdots 3 \cdot 2 \cdot 1} \cdot |x-a|^{n+1}$$

FOR ANY CONSTANT $M \geq \max_{a \leq c \leq x} f^{(n+1)}(c)$

NOTICE: $R_n(x) \to 0$

FASTER THAN

$(x-a)^n \to 0$

... WHICH IS PRETTY FAST

HW: LOOK AT ANIMATIONS ON WIKI TAYLOR SERIES