LAST TIME: HOW TO APPROXIMATE DIFF f? (I)

LINEAR APPROX AT a

$$f(x) \approx L(x) = f(a) + \frac{f'(a)(x-a)}{1}$$

↳ ERROR $|R(x)| \leq \frac{M}{2}|x-a|^2$

WHERE $M = \max_{\substack{a \leq c \leq x \\ \text{OR} \\ x \leq c \leq x}} |f''(c)|$

QUADRATIC APPROX AT a

$$f(x) \approx Q(x) = f(a) + \frac{f'(a)(x-a)}{1} + \frac{f''(a)(x-a)^2}{2}$$

n-TH TAYLOR POLYNOMIAL APPROX AT a

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2} + \frac{f^{(3)}(a)(x-a)^3}{3 \cdot 2 \cdot 1} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n(n-1)\cdots 2 \cdot 1}$$

NOTICE $P_1(x) = L(x)$ IS LIN APPROX AT a, 2013

[EXAM Q]

II

LET f BE DIFF WITH $f(3)=2$ AND $f'(3)=5$. IF TAN LINE TO GRAPH OF f AT $x=3$ IS USED TO APPROXIMATE $f(x)$, THEN AN APPROXIMATE SOLUTION FOR x TO THE EQUATION $f(x)=0$ IS

(A) $x=0.4$ (B) $x=0.5$ (C) $x=2.6$

(D) $x=3.4$ (E) $x=5.5$

$$L(x)=f(a)+f'(a)(x-a)=0$$

$$2+5(x-3)=0$$

$$5x=13$$

$$x=\frac{13}{5}\approx 2.6$$

NOT SO BAD ... IF YOU KNOW WHAT TO PLUG WHERE

RECALL WHEN IS A FUNCTION INVERTIBLE III

IF f IS ONE-TO-ONE ON DOMAIN D WITH RANGE R THEN IT HAS AN INVERSE g WITH DOMAIN R AND RANGE D

$$\begin{tikzcd} x \in D \arrow[r, bend left, "f"] & R \ni y \arrow[l, bend left, "g"] \end{tikzcd}$$

$$f(g(y)) = y$$

$$g(f(x)) = x$$

EX

$$f(x) = e^x$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left] & \mathbb{R}_{>0} \arrow[l, bend left] \end{tikzcd}$$

$$g(y) = \ln(y)$$

PROPERTIES OF f & g ARE INTER-CONNECTED

$$e^{ab} = (e^a)^b \longleftrightarrow \ln(a^b) = b\ln(a)$$

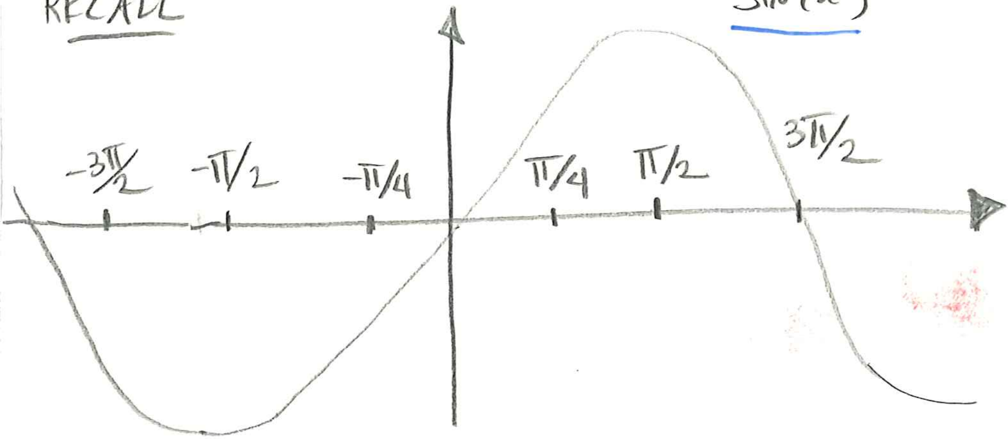
$$\frac{d}{dx}(e^x) = e^x \longleftrightarrow \frac{d}{dy}(\ln(y)) = \frac{1}{y}$$

LET'S DO SOME TRIG

(IV)

RECALL

$\sin(x)$



[Q] DOES $f(x) = \sin(x)$ HAVE AN INVERSE?

↳ YES ... IF WE RESTRICT ITS DOMAIN TO $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

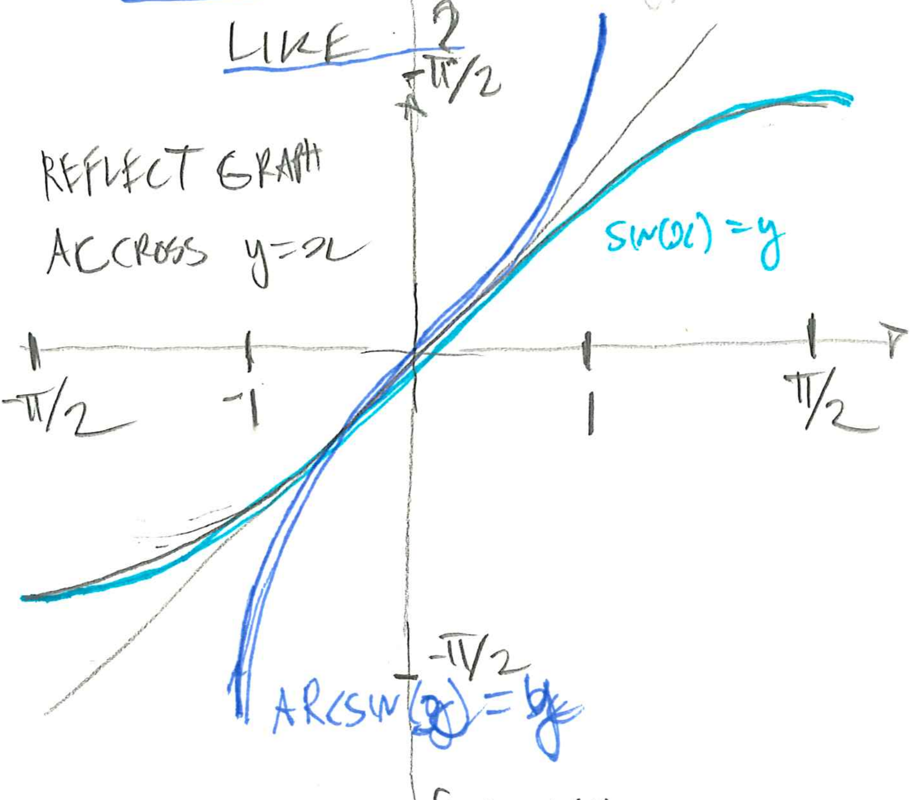
LETS CALL IT $g(y) = \arcsin(y)$

(JUST LIKE WE GAVE THE INVERSE OF $f(x) = e^x$ THE "NAME" $\ln(y)$)

↳ $\sin(\arcsin(y)) = y$

↳ $\arcsin(\sin(x)) = x$

[Q] WHAT DOES ARCSIN(y) LOOK LIKE ? (V)



$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{f(x)=\sin(x)} (-1, 1)$$

$$(-1, 1) \xrightarrow{g(y)=\arcsin(y)} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

WE KNOW $\frac{d}{dx}(\sin(x)) = \cos(x) \ldots$

CAN WE USE THIS TO FIGURE OUT $\frac{d}{dy}(\arcsin(y)) = ?$

VI

$$g(f(x)) = x$$

IMPLICIT DIFF

$$g'(f(x)) \cdot f'(x) = 1$$

$$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

RECALL $y = f(x)$

$$\Rightarrow g'(y) = \frac{1}{f'(x)}$$

OR

$$\boxed{\frac{d}{dy}(\text{ARCSIN}(y)) = \frac{1}{\cos(x)}}$$

THIS IS A FUNCTION OF $y = \sin(x)$

NEED TO MAKE INTO A FUNCTION OF y ($=\sin(x)$)

HOW!?

WHY $+\sqrt{\ }$?

$$\hookrightarrow (\sin(x))^2 + (\cos(x))^2 = 1$$

$$\Rightarrow \cos(x) = \sqrt{1 - (\sin(x))^2} = \sqrt{1 - y^2}$$

$$\Rightarrow \frac{d}{dy}(\text{ARCSIN}(y)) = \frac{1}{\sqrt{1-y^2}}$$ FOR $-1 < y < 1$

PROBLEM!?

SIMILARILY

WHERE IS COS INVERTIBLE

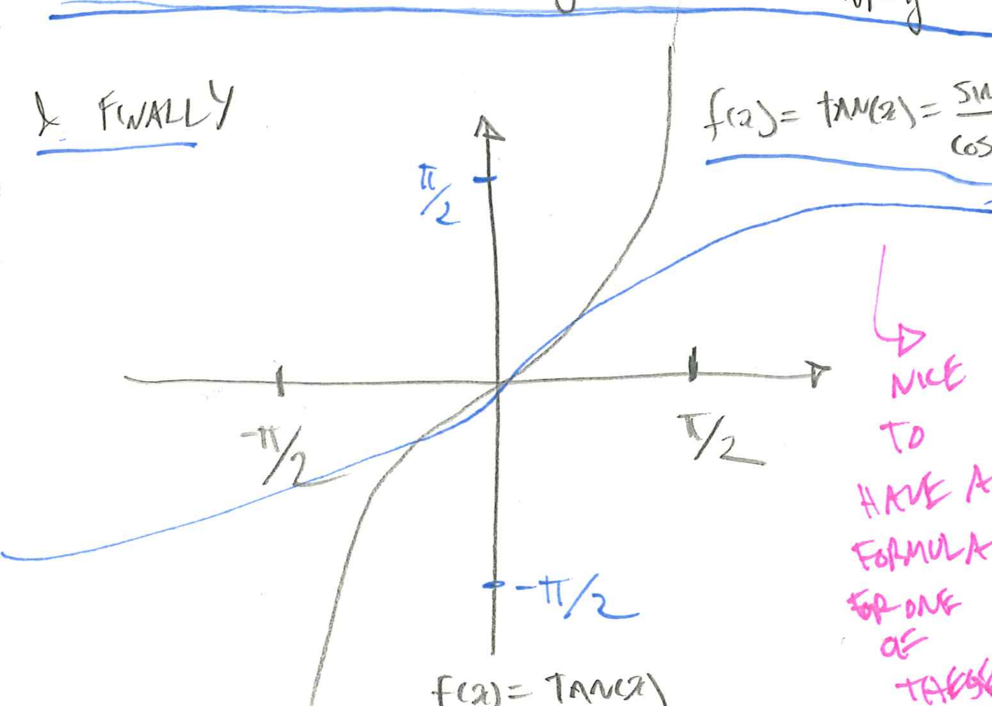
$$\begin{tikzcd} (0, \pi) \arrow[r, bend left, "f(x) = \cos(x)"] & (-1, 1) \arrow[l, bend left, "g(y) = \mathrm{ARCCOS}(y)"] \end{tikzcd}$$

WHY!?

$$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x) \longleftrightarrow \frac{d}{dy}\left(\mathrm{ARCCOS}(y)\right) = \frac{-1}{\sqrt{1-y^2}}$$

& FINALLY

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$



NICE TO HAVE A FORMULA FOR ONE OF THESE

$$\begin{tikzcd} (-\pi/2, \pi/2) \arrow[r, bend left, "f(x) = \tan(x)"] & (-\infty, +\infty) \arrow[l, bend left, "g(y) = \mathrm{TAN}(y)"] \end{tikzcd}$$

$$\boxed{\frac{d}{dx}\tan(x) = \frac{1}{\sin(x)} \Leftrightarrow \frac{d}{dy}\left(\operatorname{ATAN}(y)\right) = \frac{1}{1+y^2}}$$

VIII

↳ JUST NEED TO BE ABLE TO USE THEM AS BLACK BOXES

[EXAM Q 2012]

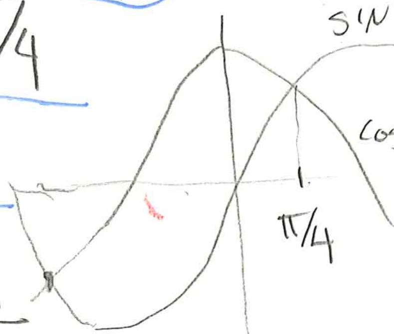
FIND THE SECOND ORDER TAYLOR POLYNOMIAL FOR $h(x) = \operatorname{ARTAN}(x)$ AT $x = 1$

$$Q(x) = h(1) + h'(1)(x-1) + \frac{h''(1)(x-1)^2}{2} \approx \operatorname{ATAN}(x)$$

$h(x) = \operatorname{ATAN}(x) \rightsquigarrow h(1) = \pi/4$

$h'(x) = \frac{1}{1+x^2} \rightsquigarrow h'(1) = \frac{1}{2}$

$h''(x) = \frac{-2x}{(1+x^2)^2} \rightsquigarrow h''(1) = \frac{-1}{2}$



$$\Rightarrow Q(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$$

NO IDEA WHAT ATAN IS BUT CAN COMPUTE SOME APPROXIMATE VALUES